Streaming Erasure Codes under Mismatched Source-Channel Frame Rates

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***Abstract*—Streaming erasure codes (*SCo*) sequentially encode a source stream into channel packets over a burst erasure channel and guarantee that each source packet is recovered within a fixed decoding delay. We study SCo codes when M channel packets need to be transmitted between successive source frames. This extends earlier works which exclusively focus on the case when $M = 1$. We obtain a general upper bound on the associated streaming capacity and show that it can be achieved for sufficiently large decoding delays using a *layered* code. For the minimum possible decoding delay we also establish the streaming capacity and show that it can be obtained using a repetition code.**

I. Introduction

In media streaming applications, a data source stream is encoded into channel packets by a transmitter. These packets are sent over a noisy channel and the receiver expects each source packet to be decoded within a fixed decoding delay. This delay constraint presents many interesting coding problems. Traditionally to protect data sent over a channel block codes such as maximum distance separable (MDS) codes have been used. These codes map blocks of data to a codeword. But such codes are not ideal in streaming applications as an entire codeword must be received before the corresponding source symbols can be decoded and the receiver cannot afford to wait for the end of the entire codeword due to delay constraints. Codes such as the digital fountain codes are also not suitable as they require the entire source data before the output stream is reproduced. Also sequential reconstruction of the source stream is not guaranteed.

In [1] (Chapter 8), [2], [3] a new class of codes called streaming erasure codes (*SCo*) is developed. Transmitter observes a source stream with one source frame arriving in each channel use. The source stream is encoded in a causal manner with rate R. The channel is modeled as a burst erasure channel in which starting at an arbitrary time, it introduces an erasure burst of maximum B channel packets. The decoder is required to output each source packet within a delay of T packets. A fundamental relationship between R, B, and T is established and *SCo* codes are constructed that achieve this tradeoff. The above setup has been extended in various directions in [4], [5], [6].

In this paper we generalize the above *SCo* setup when a total of M channel packets need to be transmitted between the arrival of successive source frames. One practical motivation

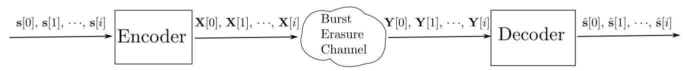

Fig. 1. System under consideration

of our model is that in many network architectures the size of a channel packet is restricted and upper layer packets are fragmented into smaller packets before transmission. Therefore the M packets between two successive source frames will be considered as a macro-packet. Note that $M = 1$ corresponds to the previously studied setup. We assume a burst-erasure channel where a burst of length B can occur across any window of B consecutive channel packets, which may span multiple macro-packets. For such a scenario we study the associated steaming capacity. We split the problem into different cases. First we obtain a general upper bound on the achievable rate in terms of M, B and T using the technique of periodic erasure channel. Then we construct codes with rates that achieve the upper bound for a certain range of parameter values. Our main code construction involves a layered approach by dividing each source frame into urgent and non-urgent source symbols and applying different levels of error correction to each to carefully form parity check symbols.

In the next section, we describe our system model in detail. We obtain the upper bound in section III. General code constructions are provided in section IV. Section V presents a special case of minimum decoding delay T (referred to as T_{min}). Finally we show an example of streaming code construction using the proposed approach in section VI followed by the conclusion.

II. System Model

We consider discrete time slots i, $i \in \mathbb{Z}$. Without loss of generality, we assume that everything before time 0 is recovered and start from $i = 0$. Figure 1 depicts the system under consideration.

A. Encoder

All symbols considered in this paper are elements from a common base field $\mathbb{F}_q$. The encoder receives a stream of i.i.d. source packets $\mathbf{s}[i]$ at the beginning of time slot i. Each $\mathbf{s}[i]$

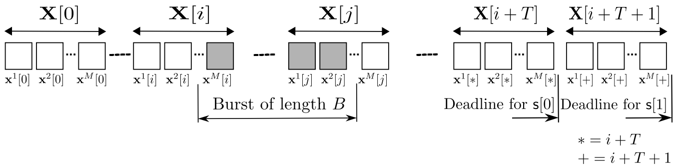

Fig. 2. General burst pattern

is causally encoded into M channel packets $\mathbf{x}^j[i] \in (\mathbb{F}_q)^{n\times 1}$, $j = [1:M]$ i.e.,

$$\mathbf{x}^j[i] = f_i^j(\mathbf{s}[i], \mathbf{s}[i-1], \cdots, \mathbf{s}[0]) \tag{1}$$

where $f_i^j(\cdot)$ is a deterministic encoding function. We call the set of these M channel packets as one macro-packet $\mathbf{X}[i] \in (\mathbb{F}_q)^{nM\times 1}$, $\mathbf{X}[i] = \left(\mathbf{x}^1[i], \mathbf{x}^2[i], \cdots, \mathbf{x}^M[i]\right)$.

B. Channel

The channel packets are sent through a burst erasure channel and received packets are denoted by $\mathbf{y}^j[i]$. The actual (channel) time slot associated with packet $\mathbf{x}^j[i]$ is $t = iM + j - 1$[1]. The channel is such that it may introduce an erasure burst of maximum B channel packets starting at an arbitrary time slot t_s and ending at $t_f \leq t_s + B - 1$. We get $\mathbf{y}^j[i] = \mathbf{x}^j[i]$, when the packet is not erased and $\mathbf{y}^j[i] = \star$, when the packet is erased i.e.,

$$\mathbf{y}^j[i] = \begin{cases} \star & \text{for } t_s \leq iM + j - 1 \leq t_f \\ \mathbf{x}^j[i] & \text{otherwise} \end{cases} \tag{2}$$

Note that the erasure burst can span multiple macro-packets as shown in Figure 2. We denote the set of all channel packets corresponding to time index i by $\mathbf{Y}[i]$, $\mathbf{Y}[i] = \left(\mathbf{y}^1[i], \mathbf{y}^2[i], \cdots, \mathbf{y}^M[i]\right)$.

C. Decoder

The decoder attempts to decode the original source stream using the received channel packets with maximum allowed delay of $T \geq 0$ macro-packets. The delay T means that the receiver must be able to decode the source packet $\mathbf{s}[i]$ at the end of macro-packet $i + T$. The decoder implements a decoding function $g_i(\cdot)$ such that,

$$\hat{\mathbf{s}}[i] = g_i(\mathbf{Y}[i+T], \mathbf{Y}[i+T-1], \cdots, \mathbf{Y}[0]) \tag{3}$$

and requires that $\hat{\mathbf{s}}[i] = \mathbf{s}[i]$.

Let $\mathsf{H}(\mathbf{s}[i]) = \mathsf{H}(\mathbf{s})$ and $\mathsf{H}(\mathbf{x}^j[i]) = \mathsf{H}(\mathbf{x})$. Rate of the streaming code is defined as the ratio of the entropy of the source packet to the size of the macro-packet i.e.,

$$R = \frac{\mathsf{H}(\mathbf{s})}{n \times M} \tag{4}$$

An optimal streaming erasure code is the one that achieves the maximum rate (i.e. *streaming capacity*) for a given choice of (M, B, T).

[1] With the exception of this section, the time index will denote macro-packet index.

III. Upper Bounds

Here we discuss an upper bound on the achievable rate of any streaming erasure code.

Theorem 1. For the given streaming setup with any M, T and B of the form $bM + B'$ where $b \in \mathbb{Z}_{\geq 0}$, $B' = [1 : M-1]$, the *streaming capacity* C is upper bounded by the following expression:

$$C \leq R^+ = 1 - \frac{B}{M(T+b+1)} \tag{5}$$

Proof: We use the technique of periodic erasure channel ([2], [4], [5]) to derive the upper bound on rate R. Consider periodic bursts each of length B with a guard interval of $M(b+T+1) - B$ as shown in Figure 3. One period length (T_{period}) is $T+b+1$ macro-packets. We let the first burst start from $\mathbf{x}^1[0]$. By definition we require all lost packets due to this erasure burst the first period to be recovered by macro-packet $t = T + b + 1$. Once these erased packets are recovered, we can treat these erasures as having never happened and simply repeat the technique for the next period and so on. Therefore we must have that the information used to recover source packets in one period must equal the information contained within the unerased channel packets in that period. We have a total of T_{period} number of source packets within one period and $(MT_{period} - B)$ number of unerased channel packets. Therefore for any streaming code with parameters (M, B, T), the following relationship must hold:

$$T_{period} \cdot \mathsf{H}(\mathbf{s}) \leq (MT_{period} - B) \cdot \mathsf{H}(\mathbf{x}) \tag{6}$$

From the definition of rate R (=$\frac{\mathsf{H}(\mathbf{s})}{M\mathsf{H}(\mathbf{x})}$), we have that for any achievable rate (and thus for *streaming capacity* C)

$$C = \max R \leq \frac{M(T+b+1) - (bM+B')}{M(T+b+1)} \tag{7}$$

which completes the proof. ■

Remark 1. In the above theorem we restrict ourselves to burst lengths B of the form $bM + B'$ ($b \in \mathbb{Z}_{\geq 0}$ and $B' = [1 : M-1]$) which leave out burst lengths $B = bM$. Bound derived above also applies to these bursts, but we can easily tighten it (using the same technique). For this case,

$$C \leq R^+ = \frac{T}{T+b} \tag{8}$$

Further we can easily show that this upper bound is achievable using the (b, T) *SCo* codes with a simple interleaving of a factor of M. So we only consider burst lengths $bM + B'$ ($b \in \mathbb{Z}_{\geq 0}$, $B' = [1 : M-1]$) and construct codes for this non-trivial case.

Remark 2. Note that we can let the first burst take other positions in the first period. However such burst positions result in loose upper bounds. The burst position considered in the proof gives the strictest bound.

Remark 3. Also by inspection of the figure 3, we observe that the minimum delay T_{min} for any positive achievable rate is b.

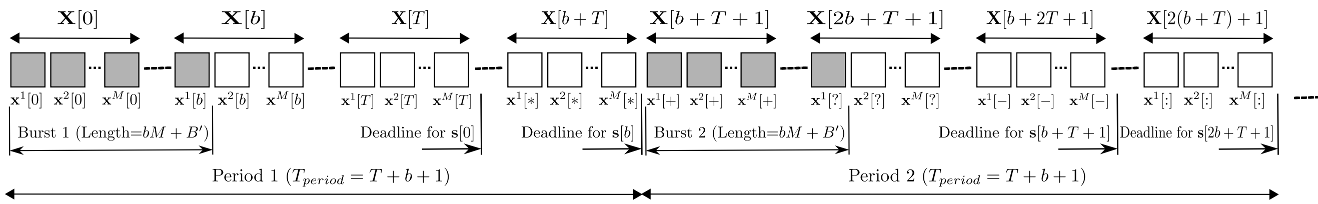

Fig. 3. Periodic erasure patterns

In the next section we describe our streaming code constructions.

IV. Code Constructions

We separately treat the case $b=0$ and $b \geq 1$ as the former special case is simpler.

A. $b=0$

If we substitute $b=0$ in the equation 5, we get the upper bound as $\frac{M(T+1)-B}{M(T+1)}$ on the streaming rate. Below we construct a code that achieves this upper bound. As pointed in the section II, all symbols are considered over $\mathbb{F}_q$.

1) Encoding: Let $k = M(T+1) - B$ and $n = T+1$. We split each source packet $\mathbf{s}[i]$ into k symbols $(s_1[i], s_2[i], \cdots, s_k[i])$. We then apply a (nM, k, T) convolutional code to form the macro-packet $\mathbf{X}[i] \in (\mathbb{F}_q)^{nM \times 1}$ such that,

$$\mathbf{X}[i] = \left(\sum_{j=0}^{T} \mathbf{s}^{\dagger}[i-j] \cdot \mathbf{G}_j \right)^{\dagger} \tag{9}$$

[2] where $\mathbf{G}_0, \cdots, \mathbf{G}_T \in (\mathbb{F}_q)^{k \times nM}$. Thus each macro-packet contains $nM = M(T+1)$ symbols. Rate R of this code is $\frac{k}{nM} = \frac{M(T+1)-B}{M(T+1)}$ as required. First T macro-packets can be expressed as

$$\begin{bmatrix} \mathbf{X}^{\dagger}[0] & \mathbf{X}^{\dagger}[1] & \cdots & \mathbf{X}^{\dagger}[T] \end{bmatrix} = \begin{bmatrix} \mathbf{s}^{\dagger}[0] & \mathbf{s}^{\dagger}[1] & \cdots & \mathbf{s}^{\dagger}[T] \end{bmatrix} \cdot \mathbf{G}_T^s \tag{10}$$

where

$$\mathbf{G}_T^s = \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \cdots & \mathbf{G}_T \\ 0 & \mathbf{G}_0 & \cdots & \mathbf{G}_{T-1} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \mathbf{G}_0 \end{bmatrix} \tag{11}$$

We consider matrices $\mathbf{G}_i$ such that they form a maximum distance profile (MDP) code [7]. Particularly we consider systematic codes where matrices $\mathbf{G}_i$ take the form

$$\mathbf{G}_0 = \begin{bmatrix} \mathbf{I}_{k \times k} & \mathbf{H}_0 \end{bmatrix}, \; \mathbf{G}_i = \begin{bmatrix} \mathbf{0}_{k \times k} & \mathbf{H}_i \end{bmatrix}, \; i = [1:T] \tag{12}$$

Matrices $\mathbf{H}_i \in (\mathbb{F}_q)^{k \times B}$ generate B parity check symbols $\mathbf{q}[i] \in (\mathbb{F}_q)^{B \times 1}$ $(q_1[i], q_2[i], \cdots, q_B[i])$ i.e.,

$$\mathbf{q}[i] = \left(\sum_{j=0}^{T} \mathbf{s}^{\dagger}[i-j] \cdot \mathbf{H}_j \right)^{\dagger} \tag{13}$$

Finally we let $\mathbf{x}^j[i] = \mathbf{X}_{(j-1)M+1}^{(j-1)M+n}[i]$.

[2] $\dagger$ denotes the transpose of a vector.

2) Decoding: We now show that the above code construction indeed corrects any erasure burst of length B within a decoding delay T. Since the code is time invariant it suffices to consider only the burst patterns that concern $\mathbf{X}[0]$ (i.e., patterns which erase symbols from $\mathbf{s}[0]$). There are maximum M such different burst patterns that need to be checked. Note that since $b=0$, $1 \leq B \leq M-1$. Thus the burst can erase symbols from at most two adjacent macro-packets. The key idea in the decoding is that in all burst positions we have enough number of unerased parity symbols (equations) to recover the lost symbols (unknowns) within decoding delay T.

Consider any burst pattern that involve $\mathbf{X}[0]$. Using [8, Corollary 3.2], it suffices to show that if we have enough parity symbols within the decoding window, we can completely recover from the erasure burst. We consider the decoding window from the macro-packet $i=0$ to the macro-packet $i=T$ with $nM(T+1)$ total symbols. We have following two cases.

- Burst is such that none of the parity symbols in the decoding window are erased. In this case, number of symbols erased=$B(T+1)$ (each channel packet contains $(T+1)$ symbols and B such channel packets are erased). Number of parity symbols within decoding window=$B(T+1)$ (each macro-packet contains B parities and none of them are erased). Hence we can completely recover within the deadline.
- Burst is such that it erases r parities. In this case, number of symbols erased=$B(T+1)-r$. Number of unerased parities within the decoding window=$B(T+1)-r$. Thus all such bursts are recoverable as well.

Remark 4. Note that the above construction simultaneously recovers all the erased packets. If the burst spans the macro-packet $i=0$ and $i=1$ then both $\mathbf{s}[0]$ and $\mathbf{s}[1]$ are simultaneously recovered at the end of the macro-packet T even though the delay of $\mathbf{s}[1]$ is later. As we will show next for $b \geq 1$, sequential recovery in general is necessary to achieve the upper bound in equation 5.

B. $b \geq 1$

Our target rate in this case is $R = \frac{M(T+b+1)-B}{M(T+b+1)}$ (upper bound).

1) Encoding: Let $k = M(T+b+1) - B$ and $n = (T+b+1)$. Each source packet is split into $k = M(T+b+1) - B$ symbols. We refer to first B symbols of each source packet as urgent symbols $\mathbf{u}[i]$ $(s_1[i], s_2[i], \cdots, s_B[i])$ and rest as non-urgent symbols (represented by $\mathbf{V}[i]$). All urgent symbols of the source packet $\mathbf{s}[i]$ are placed in the first channel packet $\mathbf{x}^1[i]$ of the macro-packet $\mathbf{X}[i]$[3]. Overall we construct a systematic (nM, k, T) convolutional code generating $(n-k) = B$ parity check symbols $\mathbf{p}[i]$ $(p_1[i], p_2[i], \cdots, p_B[i])$. The code construction involves two layers for the parity symbols. First layer is a repetition code of urgent symbols $\mathbf{u}[i]$ with a delay of T macro-packets. Second layer is a $(k-B, B, T)$ MDP code applied to the non-urgent symbols $\mathbf{V}[i]$ forming B parity symbols $\mathbf{q}[i]$ in a similar fashion as in the case of $b = 0$ (subsection IV-A) i.e.,

$$\mathbf{q}[i] = \left(\sum_{j=0}^{T} \mathbf{V}^\dagger[i-j]\mathbf{H}_j\right)^\dagger \tag{14}$$

where $\mathbf{H}_0, \cdots, \mathbf{H}_T \in (\mathbb{F})_q^{k\times B}$ and the corresponding generator matrices $\mathbf{G}_i$ form a MDP code. Thus parity symbols $\mathbf{p}[i]$ for the streaming code consist of two parts $\mathbf{p}[i] = \mathbf{q}[i] + \mathbf{u}[i-T]$, where addition is over field $\mathbb{F}_q$. We put all the parity check symbols into the last channel packet $\mathbf{x}^M[i]$ of each macro-packet[4] and divide non-urgent symbols $\mathbf{V}[i]$ to form $\mathbf{v}^1[i]$ to $\mathbf{v}^M[i]$. Overall arrangement of symbols in the macro-packet $\mathbf{X}[i]$ is as shown below.

$$\mathbf{X}[i] = \begin{bmatrix}\mathbf{u}[i]\\ \mathbf{v}^1[i]\end{bmatrix} \Bigg| \ \mathbf{v}^2[i] \ \Bigg| \ \mathbf{v}^3[i] \ \Bigg| \ \cdots \ \Bigg| \begin{bmatrix}\mathbf{v}^M[i]\\ \mathbf{p}[i]\end{bmatrix} \tag{15}$$

2) Decoding: Similar to the case of $b = 0$ (subsection IV-A), we only need to worry about M different burst patterns that erase symbols from $\mathbf{s}[0]$ (because of the time invariance of the code). Key idea here is that we recover all erased urgent symbols at their respective deadlines (that's why the name urgent) e.g., $\mathbf{u}[0]$ at T if its erased. All other erased non-urgent symbols are decoded before the deadline. Below we explicitly explain decoding for different burst positions that start at $\mathbf{x}^j[0]$.

- $j = 1$:
  - Decoding of urgent symbols - In this burst pattern, urgent symbols $\mathbf{u}[0]$ are erased. We use $\mathbf{p}[T]$ to recover them. We can do this given $\mathbf{q}[T]$ is known (which is a function of non-urgent symbols $(\mathbf{V}[0], \cdots, \mathbf{V}[T])$. Parity symbols $\mathbf{p}[j]$, $j > T$ are used to decode rest erased urgent symbols $\mathbf{u}[j-T]$.
  - Decoding of non-urgent symbols - Next we show that we can recover all the erased non-urgent symbols by macro-packet $T-1$. Using [8, Corollary 3.2] similar to the case with $b = 0$, it suffices to show that if we have enough unerased parity symbols, we can recover erased non-urgent symbols (as $\mathbf{q}$ is constructed by applying MDP to $\mathbf{V}$). We consider a window of length $(nM-B)T$ (from $\mathbf{x}^1[0]$ to $\mathbf{x}^M[T-1]$). In this window we can recover from an erasure burst of no longer than BT non-urgent symbols. Number of non-urgent symbol erased=total symbol erased-total urgent symbols erased=$B(T+b+1)-(b+1)B$=BT which is exactly what we can recover from. Thus we can recover all the erased non-urgent symbols by macro-packet $T-1$ and then at each subsequent deadline we can recover urgent symbols.
- $2 \le j \le M - B' + 1$:
  - Decoding of urgent symbols - For these burst patterns $\mathbf{x}^1[0]$(and thus $\mathbf{u}[0]$) is available at decoder. So we no longer need $\mathbf{p}[T]$ to recover $\mathbf{u}[0]$. Similar to the case above, we use parity symbols $\mathbf{p}[j]$, $j > T$ to decode rest erased urgent symbols $\mathbf{u}[j-T]$.
  - Decoding of non-urgent symbols - Since we have $\mathbf{u}[0]$, we can use $\mathbf{p}[T]$ to recover erased non-urgent symbols. Thus we consider a bigger window of length $(nM-B)(T+1)$. Therefore recoverable burst length is $B(T+1)$. Number of non-urgent symbols erased=$B(T+b+1)-bM$(since $\mathbf{u}[0]$ is no longer erased)=$B(T+1)$ which is exactly equal to the recoverable burst length.
- $j > (M - B' + 1)$:
  - Decoding of urgent symbols - These burst patterns include $\mathbf{x}^1[b+1]$. $\mathbf{x}^1[0]$ is still unerased. So recovery of urgent symbols is same as the case above.
  - Decoding of non-urgent symbols - Similar to the case above, we can use $\mathbf{p}[T]$ for the recovery of non-urgent symbols. So while the recoverable length remains the same ($B(T+1)$), the number of non-urgent erasures is B symbols less than what we have in the case above (as $\mathbf{x}^1[b+1]$ contains B urgent symbols). Hence we easily recover all the erased non-urgent symbols as required.

Remark 5. It is easy to check we can achieve a rate of $R' = \frac{M(T+1)-B}{M(T+1)}$ using random linear codes for this case. This is strictly less than the rate achieved by the construction above (as $b \ge 1$). We also see that when $b = 0$, both our codes and random linear codes achieve the same rates.

Remark 6. Note that although our model only considers a single erasure burst, as with *SCo*, our constructions correct multiple erasure-bursts separated with a sufficient guard interval.

Overall so far we have constructed codes whose rates meet the upper bound except for when $b \ge 1$ and $T_{min} \le T \le T'$. In the next section, we consider a special case of the minimum decoding delay.

V. Case with minimum decoding delay

When $T = T_{min} = b$, we obtain a tighter upper bound than that given by equation 5. We also propose a code construction to achieve the new upper bound.

[3]Here we implicitly assumed that all urgent symbols ($\mathbf{u}[i]$) are accommodated in the first channel packet ($\mathbf{x}^1[i]$). Hence we require that $B \le T+b+1$ i.e., $T \ge b(M-1) + B' - 1$ (let's call this T').

[4]For all the parity symbols ($\mathbf{p}[i]$) to be accommodated in the last channel packet ($\mathbf{x}^M[i]$), we require that $B \le T + b + 1$ i.e., $T \ge T'$ which is the same condition as the footnote above for urgent symbols.

$\mathbf{X}[0]$		$\mathbf{X}[1]$		$\mathbf{X}[0]$		$\mathbf{X}[1]$	
$\mathbf{x}^1[0]$	$\mathbf{x}^2[0]$	$\mathbf{x}^1[1]$	$\mathbf{x}^2[1]$	$\mathbf{x}^1[2]$	$\mathbf{x}^2[2]$	$\mathbf{x}^1[3]$	$\mathbf{x}^2[3]$
$u_1[0]$	$v_1^2[0]$	$u_1[1]$	$v_1^2[1]$	$u_1[2]$	$v_1^2[2]$	$u_1[3]$	$v_1^2[3]$
$u_2[0]$	$v_2^2[0]$	$u_2[1]$	$v_2^2[1]$	$u_2[2]$	$v_2^2[2]$	$u_2[3]$	$v_2^2[3]$
$u_3[0]$	$u_1[-3]+v_1^2[-3]+v_1^1[-1]$	$u_3[1]$	$u_1[-2]+v_1^2[-2]+v_1^1[0]$	$u_3[2]$	$u_1[-1]+v_1^2[-1]+v_1^1[1]$	$u_3[3]$	$u_1[0]+v_1^2[0]+v_1^1[2]$
$v_1^1[0]$	$u_2[-3]+v_2^2[-3]+v_2^1[-1]$	$v_1^1[1]$	$u_2[-2]+v_2^2[-2]+v_2^1[0]$	$v_1^1[2]$	$u_2[-1]+v_2^2[-1]+v_2^1[1]$	$v_1^1[3]$	$u_2[0]+v_2^2[0]+v_2^1[2]$
$v_2^1[0]$	$u_3[-3]+v_2^2[-2]+v_1^2[-1]$	$v_2^1[1]$	$u_3[-2]+v_2^2[-1]+v_1^2[0]$	$v_2^1[2]$	$u_3[-1]+v_2^2[0]+v_1^2[1]$	$v_2^1[3]$	$u_3[0]+v_2^2[1]+v_1^2[2]$

TABLE I
CODE CONSTRUCTION FOR $(M{=}2, B{=}3, T{=}3)$

Theorem 2. For any burst $B = bM + B'$ and delay $T = b$, the streaming capacity is given by,

$$R = \begin{cases} \frac{M-B'}{M} & B' \geq \frac{M}{2} \\ \frac{1}{2} & B' < \frac{M}{2} \end{cases} \tag{16}$$

Proof:

A. Converse

Consider a channel that erases first $B = bM + B'$ channel packets $\mathbf{x}^1[i], \ldots, \mathbf{x}^{B'}[i+b]$. Since the delay constraint for $\mathbf{s}[i]$ is $i + T = i + b$, the following equation should be satisfied,

$$\mathsf{H}(\mathbf{s}[i] | \mathbf{x}^{B'+1}[i+b], \ldots, \mathbf{x}^M[i+b]) = 0. \tag{17}$$

Now we consider a channel erasing channel packets $\mathbf{x}^{M-B'+1}[i+b], \ldots, \mathbf{x}^M[i+2b]$. The delay of $\mathbf{s}[i+b]$ is $i+2b$. Thus the following equation should be satisfied,

$$\mathsf{H}(\mathbf{s}[i+b] | \mathbf{x}^1[i+b], \ldots, \mathbf{x}^{M-B'}[i+b]) = 0. \tag{18}$$

Combining (17) and (18) one can write,

$$\mathsf{H}(\mathbf{s}[i], \mathbf{s}[i+b] | \mathbf{x}^1[i+b], \ldots, \mathbf{x}^{M-B'}[i+b], \mathbf{x}^{B'+1}[i+b], \ldots, \mathbf{x}^M[i+b]) = 0. \tag{19}$$

We have following two cases.

- If $B' \geq \frac{M}{2}$, $2(M - B')\mathsf{H}(\mathbf{x}) \geq 2\mathsf{H}(\mathbf{s})$
- If $B' < \frac{M}{2}$, $\mathsf{H}(\mathbf{x}) \geq 2\mathsf{H}(\mathbf{s})$

Therefore $R^+ = \frac{\mathsf{H}(\mathbf{s})}{M\mathsf{H}(\mathbf{x})} \leq \begin{cases} \frac{M-B'}{M} & B' \geq \frac{M}{2} \\ \frac{1}{2} & B' < \frac{M}{2} \end{cases}$ and the converse follows.

B. Achievability

For the achievability scheme, a simple repetition scheme is considered as follows:

- $B' < \frac{M}{2}$
  We split each source packet into M symbols i.e., $\mathbf{s}[i] = (s_1[i], \ldots, s_M[i])$ and assign the channel packets as follows,

$$\mathbf{x}^j[i] = \begin{cases} (s_{2j-1}[i], s_{2j}[i])^\dagger & j < \frac{M+1}{2} \\ (s_M[i], s_0[i-T])^\dagger & j^* = \frac{M+1}{2} \\ (s_{2j-M-1}[i-T], s_{2j-M}[i-T])^\dagger & j > \frac{M+1}{2} \end{cases} \tag{20}$$

  where $j = \{1, \cdots, M\}$; * - only when M is odd.
- $B' \geq \frac{M}{2}$
  In this case, we split each source packet into $M - B'$ symbols i.e., $\mathbf{s}[i] = (s_1[i], \ldots, s_{M-B'}[i])$ and assign the channel packets as follows,

$$\mathbf{x}^j[i] = \begin{cases} s_j[i] & j \in [1, M - B'] \\ 0 & j \in [M - B' + 1, B'] \\ s_{j-B'}[i - T] & j \in [B' + 1, M] \end{cases} \tag{21}$$

  where $j = \{1, \cdots, M\}$.

In each case, by inspection we can check that the codes described above are decodable within delay b. ■

Remark 7. It can easily be seen that the capacity in Theorem 2 is strictly less than the upper bound in Theorem 1.

VI. EXAMPLE

In this section we show a code construction for parameters $M = 2$, $B = 3$, $T = 3$. $B = 2 + 1$, so we have $b = 1$ and $B' = 1$. From Theorem 1, $R \leq \frac{M(T+b+1)-B}{M(T+b+1)} = \frac{7}{10}$. We construct a code with rate $\frac{7}{10}$ as follows.

A. Encoding

- Step 1: Split each source packet $\mathbf{s}[i]$ into $M(T+b+1) - B = 7$ symbols $(s_1[i], \cdots, s_7[i])$.
- Step 2: Group these into urgent and non-urgent symbols. For each source packet $\mathbf{s}[i]$, we call first $B = 3$ symbols as urgent symbols $\mathbf{u}[i]$, $(u_1[i], \cdots, u_3[i]) = (s_1[i], \cdots, s_3[i])$. Rest symbols are referred to as non-urgent symbols. We divide them to form $\mathbf{v}^1[i] = (v_1^1[i], v_2^1[i]) = (s_4[i], s_5[i])$ and $\mathbf{v}^2[i] = (v_1^2[i], v_2^2[i]) = (s_6[i], s_7[i])$. Note that in general $\mathbf{v}^1[i]$ and $\mathbf{v}^M[i]$ contain $(T + b + 1) - B$ symbols and $\mathbf{v}^j[i]$, $j = [2 : (M - 1)]$ contain $T + b + 1$ symbols as shown in equation 15.
- Step 3: We place B parity symbols $\mathbf{p}[i]$ $(p_1[i], p_2[i], p_3[i])$ into the last channel packet of each macro-packet. We generate these parities using two components, $\mathbf{p}[i]$=$\mathbf{q}[i]$+$\mathbf{u}[i-3]$. $\mathbf{q}[i]$ is formed either by applying MDP to ($\mathbf{v}^1[i]$,$\mathbf{v}^2[i]$) or more directly as $(q_1[i], q_2[i], q_3[i])$=$(v_1^2[0] + v_1^1[2], v_2^2[0] + v_2^1[2], v_2^2[1] + v_1^2[2])$. Component $\mathbf{u}[i-3]$ is just a repetition code.

B. Decoding

Since $M = 2$, there are two burst patterns that we need to check.

1) Burst that erases first three channel packets
   - Non-urgent symbol recovery - We recover $v_1^1[0]$, $v_2^1[0]$, $v_1^2[0]$ from $p_1[1]$, $p_2[1]$ $p_3[1]$ respectively and

$v_2^2[0]$, $v_1^1[1]$, $v_2^1[1]$ from $p_3[2]$, $p_1[2]$, $p_2[2]$ respectively.

- Urgent symbol recovery - Since we have recovered all the erased non-urgent symbols, we can easily recover $\mathbf{u}[0]$ from $\mathbf{p}[3]$ and $\mathbf{u}[1]$ from $\mathbf{p}[4]$ at their respective deadlines.

2) Burst that erases next three channel packets (i.e., $\mathbf{x}^2[0], \mathbf{x}^1[1], \mathbf{x}^2[1]$)

- Non-urgent symbol recovery - Since $\mathbf{u}[0]$ is unerased, we can use $\mathbf{p}[3]$ for non-urgent symbol recovery. We recover $v_1^2[0]$, $v_2^2[0]$, $v_1^1[1]$, $v_2^1[1]$ from $p_1[3]$, $p_2[3]$, $p_1[2]$, $p_2[2]$ respectively and $v_2^2[1]$ from $p_3[3]$. Finally we use previously decoded $v_2^2[0]$ to obtain $v_1^2[1]$ from $p_3[2]$.
- Urgent symbol recovery - $\mathbf{u}[0]$ is not erased and $\mathbf{u}[1]$ is easily recovered from $\mathbf{p}[4]$ since all the erased non-urgent symbols are already decoded.

VII. Conclusion

This paper studies delay constrained streaming erasure codes for mismatched source-channel frame rates where M channel packets need to be transmitted between two successive source frames. Using the technique of periodic erasure channel, a general upper bound on the streaming capacity is obtained. Streaming codes are then constructed for different cases of parameter choices. Our proposed codes are capacity achieving for certain range of parameter values (explicitly for all M, B and T such that $T \in [0, T_{min}] \cup [T', \inf)$) and achieve strictly better rates than random linear codes for any M, $B > M$ and $T \in T_{min} \cup [T', \inf)$. Finding streaming capacity for $T \in [T_{min}+1, T']$ is left for future work.

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