

Uplink-Downlink Duality Between Multiple-Access and Broadcast Channels with Compressing Relays

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Abstract—Uplink-downlink duality refers to the fact that under a sum-power constraint, the capacity regions of a Gaussian multiple-access channel and a Gaussian broadcast channel with Hermitian transposed channel matrices are identical. This paper generalizes this result to a cooperative cellular network, in which remote access-points are deployed as relays in serving the users under the coordination of a central processor (CP). In this model, the users and the relays are connected over noisy wireless links, while the relays and the CP are connected over noiseless but rate-limited fronthaul links. Based on a Lagrangian technique, this paper establishes a duality relationship between such a multiple-access relay channel and broadcast relay channel, under the assumption that the relays use compression-based strategies. Specifically, we show that under the same total transmit power constraint and individual fronthaul rate constraints, the achievable rate regions of the Gaussian multiple-access and broadcast relay channels are identical, when either independent compression or Wyner-Ziv and multivariate compression strategies are used. The key observations are that if the beamforming vectors at the relays are fixed, the sum-power minimization problems under the achievable rate and fronthaul constraints in both the uplink and the downlink can be transformed into either a linear programming or a semidefinite programming problem depending on the compression technique, and that the uplink and downlink problems are Lagrangian duals of each other. Moreover, the dual variables corresponding to the downlink rate constraints become the uplink powers; the dual variables corresponding to the downlink fronthaul constraints become the uplink quantization noises. This duality relationship enables an efficient algorithm for optimizing the downlink transmission and relaying strategies based on the uplink.

Index Terms—Uplink-downlink duality, multiple-access channel, broadcast channel, relay channel, Wyner-Ziv compression, multivariate compression, cloud radio access network, cell-free massive multiple-input multiple-output (MIMO).

I. INTRODUCTION

There is a curious uplink-downlink duality between the Gaussian multiple-access channel with a multiple-antenna receiver and the Gaussian broadcast channel with a multiple-antenna transmitter — under the same total power constraint,

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the uplink and downlink achievable rate regions with linear processing, or alternatively the uplink and downlink capacity regions with optimal nonlinear processing, are identical [1]. While the traditional multiple-access and broadcast channel models are suited for an isolated wireless cellular system with a single base-station (BS), in this paper, we are motivated by a generalization of this model to cooperative cellular networks in which BSs cooperate over *rate-limited* digital links to a central processor (CP) in communicating with the users. In this model, the BSs in effect act as remote radio-heads with finite-capacity fronthaul links and function as relays between the CP and the users. The aim of this paper is to establish an uplink-downlink duality between achievable rate regions of such a Gaussian multiple-access *relay* channel and a Gaussian broadcast *relay* channel.

The centralized cooperative communication architecture, in which multiple relay-like BSs cooperatively serve the users under the coordination of a CP, is an appealing solution to the ever-increasing demand for mobile broadband in future wireless communication networks, because of its ability to mitigate intercell interference. Under the above architecture, the users and the relay-like BSs are connected by noisy wireless channels, while the relay-like BSs and the CP are connected by noiseless fronthaul links of finite rate limit, as shown in Fig. 1. In the uplink, the users transmit their signals and the relay-like BSs forward their received signals to the CP for joint information decoding, while in the downlink, the CP jointly encodes the user messages and sends them to the relay-like BSs to transmit to the users. Because the CP can jointly decode and encode the user messages, this cooperative architecture can effectively utilize cross-cell links to enhance message transmission, instead of treating signals from neighboring cells as interference, thus enabling a significant improvement in the overall network throughput. In the literature, there are several terminologies to describe the above centralized cellular architecture, including coordinated multipoint (CoMP) [2], distributed antenna system (DAS) [3], cloud radio access network (C-RAN) [4], cell-free massive multiple-input multiple-output (MIMO) [5], etc. All of these systems can be modeled by the multiple-access relay channel in the uplink and the broadcast relay channel in the downlink as discussed above, where the hop between the users and the relays is wireless, while the hop between the relays and the CP is digital.

When the capacities of all the fronthaul links are infinite, the above model reduces to the traditional multiple-access and broadcast channels. In this case, the relays can simply be treated as remote antennas of a single virtual BS over

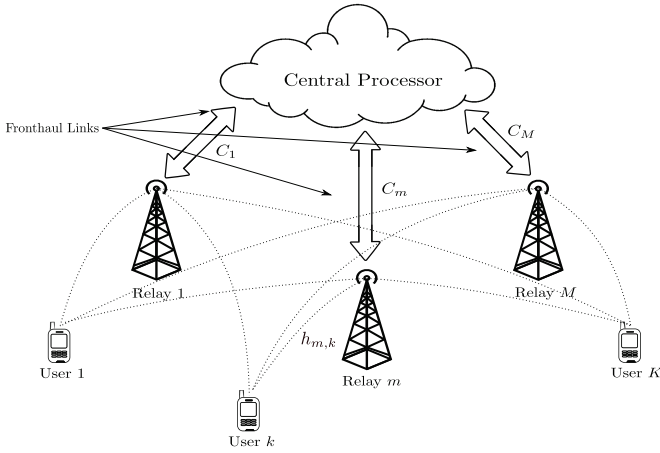


Fig. 1. Cooperative cellular network in which the relay-like BSs are connected to the CP via rate-limited fronthaul links and the user messages are jointly encoded/decoded at the CP.

the entire network. From the existing literature, we know that there exists a duality relationship between the multiple-access channel and the broadcast channel [1], [6]–[12], which states that given the same sum-power constraint, any rate-tuple achievable in the uplink is also achievable in the downlink, and vice versa. This duality holds under both linear and nonlinear receiving/precoding at the BS. We now ask the following question: If the capacity of the fronthaul links is finite, does a similar uplink-downlink duality relationship hold? The answer to this question depends on not only the joint processing scheme at the CP, but also the relaying strategies employed at the BSs. In practical implementations, a variety of ways of jointly optimizing the utilization of fronthaul and the wireless links have been proposed, e.g., for CoMP [13], [14], DAS [15], C-RAN [16]–[21], and cell-free massive MIMO [22]–[24]. This paper provides an affirmative answer to the above question in the sense that if the compression-based strategies are utilized over the fronthaul links, then indeed the achievable rate regions of the Gaussian multiple-access relay channel and the Gaussian broadcast relay channel are identical under the same sum-power constraint and individual fronthaul capacity constraints. This duality relationship holds with either independent compression [16], [19], [20], [25] or Wyner-Ziv/multivariate compression [16]–[18], [21] at the relays, and with either linear or nonlinear processing at the CP.

A. Prior Works on Uplink-Downlink Duality

1) *Linear Encoding and Linear Decoding*: The uplink-downlink duality between the multiple-access channel and the broadcast channel is first established in the case of linear encoding/decoding while treating interference as noise. Assuming single-antenna users and a multiple-antenna BS, the main result is that any signal-to-interference-plus-noise ratio (SINR) tuple that is achievable in the multiple-access channel can also be achieved in the broadcast channel under the same sum-power constraint, and vice versa. This duality has been proved in [6]–[9], [11] as follows. First, it is shown that if the receive beamforming vectors in the multiple-access

channel and the transmit beamforming vectors in the broadcast channel are identical, then the feasibility conditions to ensure that an SINR-tuple can be achieved for both the multiple-access and the broadcast channels via power control are the same. The key observation is that a feasible power control solution exists if and only if the spectral radius of a so-called interference matrix is less than one [26], while given the same receive/transmit beamforming vectors and SINR targets, the spectral radii of the interference matrices of the multiple-access and the broadcast channels are the same. Moreover, it is shown that given the same transmit/receive beamforming vectors, the minimum total transmit power to achieve a set of feasible SINR targets in the uplink is the same as that to achieve the same set of SINR targets in the downlink. As a result, the achievable SINR regions of the multiple-access and broadcast channels are identical under the same power constraint.

An alternative approach to proving the above duality result for the case of linear encoding and linear decoding is based on the Lagrangian duality technique [10]. Specifically, given the receive/transmit beamforming vectors, the power control problems of minimizing the total transmit power subject to the users' individual SINR constraints in the multiple-access channel and the broadcast channel are both convex. Moreover, if the receive beamforming vectors and users' SINR targets in the multiple-access channel are the same as the transmit beamforming vectors and users' SINR targets in the broadcast channel, then the Lagrangian dual of the uplink sum-power minimization problem can be shown to be equivalent to the downlink sum-power minimization problem, and vice versa. This shows that the achievable SINR regions of the multiple-access channel and the broadcast channel are identical under the same sum-power constraint.

2) *Nonlinear Encoding and Nonlinear Decoding*: The uplink-downlink duality between the multiple-access channel and the broadcast channel is also established in the case of nonlinear encoding/decoding. Assuming again the case of single-antenna users, the main result is that the sum capacity (and also the capacity region) of the multiple-access channel, which is achieved by successive interference cancellation [27], is the same as the sum capacity (or the capacity region) of the broadcast channel, which is achieved by dirty-paper coding [28]. Similar to the linear encoding/decoding case [6]–[9], it is shown in [11] that if the decoding order for successive interference cancellation is the reverse of the encoding order for dirty-paper coding, and the uplink receive beamforming vectors are the same as the downlink transmit beamforming vectors, then the feasibility conditions to ensure that an SINR-tuple can be achieved in both the multiple-access channel and the broadcast channel via power control are the same. Moreover, it is shown in [11] that if each user achieves the same SINR in the multiple-access channel and the broadcast channel, then the total transmit power in the uplink is the same as in the downlink. As a result, under the same sum-power constraint, the capacity region of the multiple-access channel is the same as the capacity region of the broadcast channel achieved by dirty-paper coding. This duality result can be extended [1] even to the case where the users are equipped

with multiple antennas, using a clever choice of the transmit covariance matrix for the broadcast channel to achieve each achievable rate-tuple in the multiple-access channel and vice versa.

For the sum-capacity problem, there is also an alternative approach of establishing uplink-downlink duality based on the Lagrangian duality of a minimax problem. Along this line, [12] shows that the sum capacity of the broadcast channel can be characterized by a minimax optimization problem, where the maximization is over the transmit covariance matrix, while the minimization is over the receive covariance matrix. Since this minimax problem is a convex problem, it is equivalent to its dual problem, which is another minimax problem in the multiple-access channel. Moreover, the optimal value of this new minimax problem is shown to be exactly the sum capacity of the multiple-access channel. As a result, the sum capacities of the multiple-access channel and the broadcast channel are the same.

3) *Other Duality Results:* Apart from the above results, the duality relationship is also established for the multiple-access channel and the broadcast channel under different power constraints and encoding/decoding strategies. For example, [29] shows that the power minimization problem in the broadcast channel under the per-antenna power constraints is equivalent to the minimax problem in the multiple-access channel with an uncertain noise. Moreover, [30] shows that the rate region of the broadcast channel achieved by dirty-paper coding and under multiple power constraints is the same as that of the multiple-access channel achieved by successive interference cancellation and under a weighted sum-power constraint. This result is generalized in [31], [32] to the uplink and downlink interference networks under multiple linear constraints. Further, [33] shows that any sum rate achievable via integer-forcing in the MIMO multiple-access channel can be achieved via integer-forcing in the MIMO broadcast channel with the same sum-power and vice versa. In [34], the duality between the multiple-access channel and the broadcast channel is extended to the scenario with a full-duplex BS. Moreover, the duality relationship is also investigated for the multiple-access channel and the broadcast channel with amplify-and-forward relays. It is shown in [35]–[37] that for both the two-hop and multihop relay scenarios, the user rate regions are the same in the uplink and downlink under the same sum-power constraint. This result can also be generalized to cases with multiple linear constraints [38]. Finally, duality is used in [39] to characterize the polynomial-time solvability of a power control problem in the multiple-input single-output (MISO) and single-input multiple-output (SIMO) interference channels.

B. Overview of Main Results

This paper establishes a duality relationship between the multiple-access relay channel and the broadcast relay channel when the compression-based relay strategies are used over the rate-limited fronthaul links between the CP and the relays. Both the users and the relays are assumed to be equipped with a single antenna. In the uplink, each relay compresses its received signals from the users, and sends the compressed signal

to the CP via the fronthaul link. The CP first decompresses the signals from the relays, then jointly decodes the user messages based on the decompressed signals. In the downlink, the CP jointly encodes the user messages, compresses the transmit signals for the relays, and sends the compressed signals to the relays via the fronthaul links. Then, each relay decompresses its received signal and transmits it to the users. Compared to the classic uplink-downlink duality results in the literature, the novel contributions of our work are as follows.

We show that under the same sum-power constraint and individual fronthaul capacity constraints, the achievable rate regions of the multiple-access channel and the broadcast channel are identical using compression-based relays, under the following four cases:

- I: In the uplink, the relays compress their received signals independently and the CP applies the linear decoding strategy. In the downlink, the CP applies the linear encoding strategy and compresses the signals for the relays independently.
- II: In the uplink, the relays compress their received signals independently and the CP applies the successive interference cancellation strategy. In the downlink, the CP applies the dirty-paper coding strategy and compresses the signals for the relays independently.
- III: In the uplink, the relays apply the Wyner-Ziv compression strategy to compress their received signals and the CP applies the linear decoding strategy. In the downlink, the CP applies the linear encoding strategy and the multivariate compression strategy to compress the signals for the relays.
- IV: In the uplink, the relays apply the Wyner-Ziv compression strategy to compress their received signals and the CP applies the successive interference cancellation strategy. In the downlink, the CP applies the dirty-paper coding strategy and the multivariate compression strategy to compress the signals for the relays.

Note that the conventional uplink-downlink duality relationship between the multiple-access channel and the broadcast channel [1], [6]–[12] is a special case of the duality relationship established in this work if we assume the fronthaul links all have infinite capacities.

For Cases I and II with independent compression for the relays, we provide an alternative proof for the duality relationship as compared to our previous work [40]. Specifically, the duality relationship is validated based on the Lagrangian duality [41], which provides a unified approach for all the cases. In particular, we show that given the transmit beamforming vectors in the downlink, the sum-power minimization problem subject to the individual user rate constraints and individual fronthaul capacity constraints is a linear program (LP) with strong duality. Then, it is shown that given the same receive and transmit beamforming vectors (and the reversed decoding order and encoding order for Case II) and under the same individual user rate constraints as well as individual fronthaul capacity constraints, the uplink sum-power minimization problem is equivalent to the Lagrangian dual of the downlink sum-power minimization problem. This approach is similar

to that used in [10] to verify the duality of the conventional multiple-access channel and broadcast channel without relays. However, interesting new insights can be obtained when relays are deployed between the CP and the users. Specifically, the dual variables associated with the user rate constraints and the fronthaul capacity constraints in the downlink sum-power minimization problem play the role of uplink user transmit powers and uplink relay quantization noise levels, respectively, in the dual problem.

For Cases III and IV, we establish a novel duality relationship between Wyner-Ziv compression and multivariate compression [42]. Intuitively, Wyner-Ziv compression over the noiseless fronthaul resembles successive interference cancellation in the noisy wireless channel in the sense that the decompressed signals can provide side information for decompressing the remaining signals. On the other hand, multivariate compression in the noiseless fronthaul resembles dirty-paper coding in the noisy wireless channel in the sense that it can control the interference caused by compression seen by the users. Despite the well-known duality between successive interference cancellation and dirty-paper coding, the relationship between these two compression strategies has not been established previously. We show in this work that if the decompression order in Wyner-Ziv compression is the reverse of the compression order in the multivariate compression, the uplink-downlink duality remains true between the multiple-access relay channel and the broadcast relay channel.

To prove the above result, we use the Lagrangian duality approach similar to that taken in [10], [12], rather than the approach of checking the feasibility conditions adopted in [6]–[9], [11]. This is because under the Wyner-Ziv and multivariate compression strategies, the fronthaul rates are complicated functions of the transmit powers and quantization noises. In this case, the fronthaul capacity constraints are no longer linear in the variables, and the feasibility condition proposed in [26] for linear constraints does not work. Despite the complicated fronthaul rate expressions, we reveal that under the multivariate compression strategy in the broadcast relay channel, if the transmit beamforming vectors are fixed, the sum-power minimization problem subject to individual user rate constraints and individual fronthaul capacity constraints can be transformed into a convex optimization problem over the transmit powers as well as the compression noise covariance matrix. Then, we characterize the dual problem of this convex optimization. It turns out that if we interpret the dual variables associated with the user rate constraints as the uplink transmit powers and some diagonal elements of the dual matrices associated with the fronthaul capacity constraints as the uplink compression noise levels, then the Lagrangian dual of the broadcast relay channel problem is equivalent to the sum-power minimization problem in the multiple-access relay channel subject to individual user rate constraints as well as a single matrix inequality constraint. In contrast to Cases I and II with independent compression where the primal downlink problem is an LP and its dual problem directly has individual fronthaul capacity constraints, here the problem is a semidefinite program (SDP), and we need to take a further step to transform the single matrix inequality constraint into

the individual fronthaul capacity constraints under the Wyner-Ziv compression strategy via a series of matrix operations. At the end, we show that at the optimal solution, all the user rate constraints and fronthaul capacity constraints in the dual problem are satisfied with equality, and moreover there exists a unique solution to this set of nonlinear equations. As a result, the dual of the broadcast relay channel problem is equivalent to the multiple-access relay channel problem.

C. Organization

The rest of this paper is organized as follows. Section II describes the system model and characterizes the achievable rate regions of the multiple-access relay channel and the broadcast relay channel under various encoding/decoding and compression/decompression strategies. In Section III, the duality relationship between the multiple-access channel and the broadcast relay channel with compression-based relays is established. Sections IV to VII prove the duality for Cases I-IV, respectively, based on the Lagrangian duality theory. We summarize the main duality relations in Section VIII and provide an application of duality in optimizing the broadcast relay channel via its dual multiple-access relay channel in Section IX. We conclude this paper with Section X.

Notation: Scalars are denoted by lower-case letters; vectors are denoted by bold-face lower-case letters; matrices are denoted by bold-face upper-case letters. We use \mathbf{I} to denote an identity matrix with an appropriate dimension, $\mathbf{0}_{x \times y}$ to denote an all-zero matrix with dimension of $x \times y$. For a matrix \mathbf{A} , $\mathbf{A}^{(x,y)}$ denotes the entry on the x -th row and the y -th column of \mathbf{A} , and $\mathbf{A}^{(x_1:y_1, x_2:y_2)}$ denotes a submatrix of \mathbf{A} defined by

$$\mathbf{A}^{(x_1:y_1, x_2:y_2)} = \begin{bmatrix} \mathbf{A}^{(x_1, y_1)} & \dots & \mathbf{A}^{(x_1, y_2)} \\ \vdots & \ddots & \vdots \\ \mathbf{A}^{(x_2, y_1)} & \dots & \mathbf{A}^{(x_2, y_2)} \end{bmatrix}.$$

For a square full-rank matrix \mathbf{S} , \mathbf{S}^{-1} denotes its inverse, and $\mathbf{S} \succeq \mathbf{0}$ or $\mathbf{S} \succ \mathbf{0}$ indicates that \mathbf{S} is a positive semidefinite matrix or a positive definite matrix, respectively. For a matrix \mathbf{M} of an arbitrary size, \mathbf{M}^H , \mathbf{M}^T , and \mathbf{M}^* denote the conjugate transpose, transpose and conjugate of \mathbf{M} , respectively, and $\text{rank}(\mathbf{M})$ denotes the rank of \mathbf{M} . We use $\text{diag}(x_1, \dots, x_K)$ to denote a diagonal matrix with the diagonal elements given by x_1, \dots, x_K . For two real vectors $\mathbf{x} = [x_1, \dots, x_N]^T$ and $\mathbf{y} = [y_1, \dots, y_N]^T$, $\mathbf{x} \geq \mathbf{y}$ means that $x_n \geq y_n, \forall n = 1, \dots, N$.

II. SYSTEM MODEL AND ACHIEVABLE RATE REGIONS

The Gaussian multiple-access relay channel and the Gaussian broadcast relay channel considered in this paper consist of one CP, M single-antenna relays, denoted by the set $\mathcal{M} = \{1, \dots, M\}$, and K single-antenna users, denoted by the set $\mathcal{K} = \{1, \dots, K\}$, as shown in Fig. 2, where each relay $m \in \mathcal{M}$ is connected to the users over the wireless channels and to the CP via the noiseless digital fronthaul link of capacity C_m bits per symbol (bps). For the multiple-access

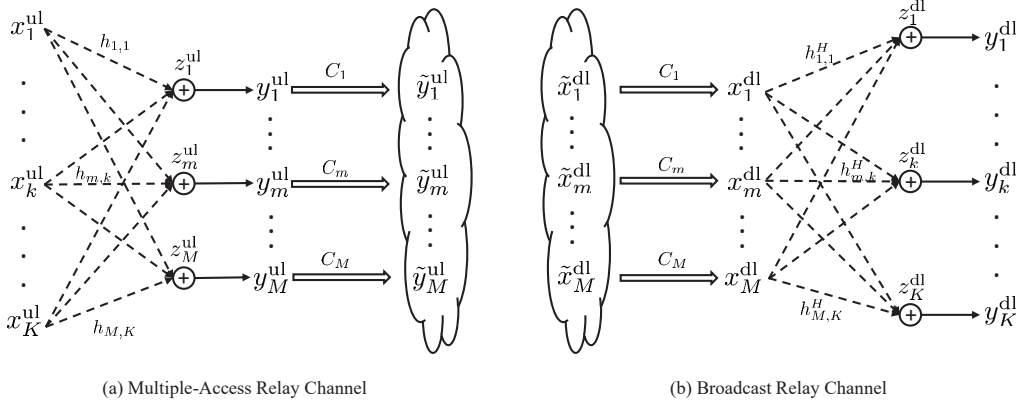


Fig. 2. System model of the multiple-access relay channel and the broadcast relay channel.

relay channel, the overall channel from user $k \in \mathcal{K}$ to all the relays is denoted by

$$\mathbf{h}_k = [h_{1,k}, \dots, h_{M,k}]^T, \quad \forall k \in \mathcal{K}, \quad (1)$$

where $h_{m,k}$ denotes the channel from user $k \in \mathcal{K}$ to relay $m \in \mathcal{M}$; in the dual broadcast relay channel, the overall channel from all the relays to user $k \in \mathcal{K}$ is the Hermitian transpose of the corresponding uplink channel, i.e., \mathbf{h}_k^H . Further, we assume a sum-power constraint P for all the users in the multiple-access relay channel, and the same sum-power constraint P for all the relays in the broadcast relay channel. In the following, we review the compression-based relaying strategies [4], [43] for the multiple-access relay channel and the broadcast relay channel in detail. These compression-based strategies are simplified versions of the more general relaying strategies for the multihop relay networks studied in [44], [45]. In particular, the compression-based strategies considered in this paper take the approach of separating the encoding/decoding of the relay codeword and the encoding/decoding of the user messages, in contrast to the joint encoding/decoding approach in [44], [45]. In several specific cases [17], [46], these simplified strategies can be shown to already achieve the capacity regions of the specific Gaussian multiple-access and broadcast relay channels to within a constant gap.

A. Multiple-Access Channel with Compressing Relays

The Gaussian multiple-access relay channel model is as shown in Fig. 2(a). The discrete-time baseband channel between the users and the relays can be modelled as

$$\begin{bmatrix} y_1^{\text{ul}} \\ \vdots \\ y_M^{\text{ul}} \end{bmatrix} = \begin{bmatrix} h_{1,1} & \cdots & h_{1,K} \\ \vdots & \ddots & \vdots \\ h_{M,1} & \cdots & h_{M,K} \end{bmatrix} \begin{bmatrix} x_1^{\text{ul}} \\ \vdots \\ x_K^{\text{ul}} \end{bmatrix} + \begin{bmatrix} z_1^{\text{ul}} \\ \vdots \\ z_M^{\text{ul}} \end{bmatrix}, \quad (2)$$

where x_k^{ul} denotes the transmit signal of user k , $\forall k \in \mathcal{K}$, and $z_m^{\text{ul}} \sim \mathcal{CN}(0, \sigma^2)$ denotes the additive white Gaussian noise (AWGN) at relay m , $\forall m \in \mathcal{M}$.

Transmission and relaying strategies for the multiple-access relay channel have been studied extensively in the literature [14], [16]–[18], [47]. In this paper, we assume the following

strategy in which each user transmits using a Gaussian codebook, i.e.,

$$x_k^{\text{ul}} = \sqrt{p_k^{\text{ul}}} s_k^{\text{ul}}, \quad \forall k \in \mathcal{K}, \quad (3)$$

where $s_k^{\text{ul}} \sim \mathcal{CN}(0, 1)$ denotes the message of user k , and $p_k^{\text{ul}} \geq 0$ denotes the transmit power of user k . As a result, the total transmit power of all the users is expressed as

$$P^{\text{ul}}(\{p_k^{\text{ul}}\}) = \sum_{k=1}^K \mathbb{E}[|x_k^{\text{ul}}|^2] = \sum_{k=1}^K p_k^{\text{ul}}. \quad (4)$$

After receiving the wireless signals from the users, we assume that relay m compresses y_m^{ul} and sends the compressed signals to the CP, $\forall m$. We assume a Gaussian compression codebook and model the quantization noise introduced in the compression process as an independent Gaussian random variable, i.e,

$$\tilde{y}_m^{\text{ul}} = y_m^{\text{ul}} + e_m^{\text{ul}} = \sum_{k=1}^K h_{m,k} x_k^{\text{ul}} + z_m^{\text{ul}} + e_m^{\text{ul}}, \quad \forall m \in \mathcal{M}, \quad (5)$$

where $e_m^{\text{ul}} \sim \mathcal{CN}(0, q_m^{\text{ul}})$ denotes the compression noise at relay m , with $q_m^{\text{ul}} \geq 0$ denoting its variance. While the Gaussian compression codebook is not necessarily optimal [47], it gives tractable achievable rate regions. Note that e_m^{ul} 's are independent of y_m^{ul} 's and are independent across m . In other words, if we define $\mathbf{e}^{\text{ul}} = [e_1^{\text{ul}}, \dots, e_M^{\text{ul}}]^T$, then it follows that

$$\mathbb{E}[\mathbf{e}^{\text{ul}} (\mathbf{e}^{\text{ul}})^H] = \text{diag}(q_1^{\text{ul}}, \dots, q_M^{\text{ul}}) \succeq \mathbf{0}. \quad (6)$$

After receiving the compressed signals, the CP first decodes the compression codewords then decodes each user's message based on the beamformed signals, i.e.,

$$\tilde{s}_k^{\text{ul}} = \mathbf{w}_k^H \tilde{\mathbf{y}}^{\text{ul}}, \quad \forall k \in \mathcal{K}, \quad (7)$$

where $\mathbf{w}_k = [w_{k,1}, \dots, w_{k,M}]^T$ with $\|\mathbf{w}_k\|^2 = 1$ denotes the receive beamforming vector for user k 's message, and $\tilde{\mathbf{y}}^{\text{ul}} = [\tilde{y}_1^{\text{ul}}, \dots, \tilde{y}_M^{\text{ul}}]^T$ denotes the collective compressed signals from all the relays.

$$\begin{aligned}
C_{\rho^{\text{ul}}(m)}^{\text{ul,WZ}}(\{p_k^{\text{ul}}\}, q_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, q_{\rho^{\text{ul}}(m)}^{\text{ul}}, \{\rho^{\text{ul}}(m)\}) &= I(\hat{y}_{\rho^{\text{ul}}(m)}^{\text{ul}}; y_{\rho^{\text{ul}}(m)}^{\text{ul}} | \hat{y}_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, \hat{y}_{\rho^{\text{ul}}(m-1)}^{\text{ul}}) \\
&= \log_2 \frac{|\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(1:m,1:m)}|}{|\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(1:m-1,1:m-1)}| q_{\rho^{\text{ul}}(m)}^{\text{ul}}} \\
&= \log_2 \frac{\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(m,m)} - \mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(m,1:m-1)} (\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(1:m-1,1:m-1)})^{-1} \mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}^{(1:m-1,m)}}{q_{\rho^{\text{ul}}(m)}^{\text{ul}}}, \quad \forall m \in \mathcal{M},
\end{aligned} \tag{9}$$

where

$$\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}} = \mathbb{E} \left[\tilde{\mathbf{y}}_{\{\rho^{\text{ul}}(m)\}}^{\text{ul}} \left(\tilde{\mathbf{y}}_{\{\rho^{\text{ul}}(m)\}}^{\text{ul}} \right)^H \right] = \sum_{k=1}^K p_k^{\text{ul}} \bar{\mathbf{h}}_{k,\{\rho^{\text{ul}}(m)\}} (\bar{\mathbf{h}}_{k,\{\rho^{\text{ul}}(m)\}})^H + \sigma^2 \mathbf{I} + \text{diag}(q_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, q_{\rho^{\text{ul}}(M)}^{\text{ul}}), \tag{10}$$

$$\tilde{\mathbf{y}}_{\{\rho^{\text{ul}}(m)\}}^{\text{ul}} = [\tilde{y}_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, \tilde{y}_{\rho^{\text{ul}}(M)}^{\text{ul}}]^T, \tag{11}$$

$$\bar{\mathbf{h}}_{k,\{\rho^{\text{ul}}(m)\}} = [h_{\rho^{\text{ul}}(1),k}, \dots, h_{\rho^{\text{ul}}(M),k}]^T, \quad \forall k \in \mathcal{K}. \tag{12}$$

$$\begin{aligned}
R_{\tau^{\text{ul}}(k)}^{\text{ul,SIC}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}, \{\tau^{\text{ul}}(k)\}) &= I(s_{\tau^{\text{ul}}(k)}^{\text{ul}}; \tilde{s}_{\tau^{\text{ul}}(k)}^{\text{ul}} | s_{\tau^{\text{ul}}(1)}^{\text{ul}}, \dots, s_{\tau^{\text{ul}}(k-1)}^{\text{ul}}) \\
&= \log_2 \frac{\sum_{i \geq k} p_{\tau^{\text{ul}}(i)}^{\text{ul}} |\mathbf{w}_{\tau^{\text{ul}}(k)}^H \mathbf{h}_{\tau^{\text{ul}}(i)}|^2 + \sum_{m=1}^M q_m^{\text{ul}} |w_{\tau^{\text{ul}}(k),m}|^2 + \sigma^2}{\sum_{j > k} p_{\tau^{\text{ul}}(j)}^{\text{ul}} |\mathbf{w}_{\tau^{\text{ul}}(k)}^H \mathbf{h}_{\tau^{\text{ul}}(j)}|^2 + \sum_{m=1}^M q_m^{\text{ul}} |w_{\tau^{\text{ul}}(k),m}|^2 + \sigma^2}, \quad \forall k \in \mathcal{K}.
\end{aligned} \tag{14}$$

1) *Compression Strategies:* We consider two compression strategies at the relays in this work: the independent compression strategy and the Wyner-Ziv compression strategy. If independent compression is performed across the relays, the fronthaul rate for transmitting \tilde{y}_m^{ul} is expressed as

$$\begin{aligned}
C_m^{\text{ul,IN}}(\{p_k^{\text{ul}}\}, q_m^{\text{ul}}) &= I(y_m^{\text{ul}}; \tilde{y}_m^{\text{ul}}) \\
&= \log_2 \frac{\sum_{k=1}^K p_k^{\text{ul}} |h_{m,k}|^2 + q_m^{\text{ul}} + \sigma^2}{q_m^{\text{ul}}}, \quad \forall m \in \mathcal{M}.
\end{aligned} \tag{8}$$

Alternatively, the relays can also perform the Wyner-Ziv compression strategy in a successive fashion, accounting for the fact that the compressed signals are to be decoded jointly at the CP. Given a decompression order $\rho^{\text{ul}}(1), \dots, \rho^{\text{ul}}(M)$ at the CP in which the signal from relay $\rho^{\text{ul}}(1) \in \mathcal{M}$ is decompressed first, the signal from relay $\rho^{\text{ul}}(2) \in \mathcal{M}$ is decoded second (with $\rho^{\text{ul}}(1)$ as side information), etc., the Wyner-Ziv compression rate [4] of relay $\rho^{\text{ul}}(m)$ can be expressed as (9)–(12) on top of the page.

In words, $\bar{\mathbf{h}}_{k,\{\rho^{\text{ul}}(m)\}}$ denotes the collection of the channels from user k to relays $\rho^{\text{ul}}(1), \dots, \rho^{\text{ul}}(M)$, $\tilde{\mathbf{y}}_{\{\rho^{\text{ul}}(m)\}}^{\text{ul}}$ is the collection of the compressed received signals from relay $\rho^{\text{ul}}(1)$ to relay $\rho^{\text{ul}}(M)$, and $\mathbf{\Gamma}_{\{\rho^{\text{ul}}(m)\}}$ is the covariance matrix of this re-ordered vector.

2) *Decoding Strategies:* We consider two decoding strategies at the CP: the linear decoding strategy by treating interference as noise and the nonlinear decoding strategy with successive interference cancellation. First, if the CP treats interference as noise, the achievable rate of user k is expressed

as

$$\begin{aligned}
R_k^{\text{ul,TIN}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) &= I(s_k^{\text{ul}}; \tilde{s}_k^{\text{ul}}) \\
&= \log_2 \frac{\sum_{i=1}^K p_i^{\text{ul}} |\mathbf{w}_k^H \mathbf{h}_i|^2 + \sum_{m=1}^M q_m^{\text{ul}} |w_{k,m}|^2 + \sigma^2}{\sum_{j \neq k} p_j^{\text{ul}} |\mathbf{w}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M q_m^{\text{ul}} |w_{k,m}|^2 + \sigma^2}, \quad \forall k \in \mathcal{K}.
\end{aligned} \tag{13}$$

If the successive interference cancellation strategy is used, given a decoding order $\tau^{\text{ul}}(1), \dots, \tau^{\text{ul}}(K)$ at the CP in which the message of user $\tau^{\text{ul}}(1)$ is decoded first, the message of user $\tau^{\text{ul}}(2)$ is decoded second, etc., the achievable rate of user $\tau^{\text{ul}}(k)$ is expressed as (14) on top of the page.

3) *Achievable Rate Regions:* Given the individual fronthaul capacity constraints C_m 's and sum-power constraint P , define $\mathcal{T}^{\text{ul,IN}}(\{C_m\}, P)$ and $\mathcal{T}^{\text{ul,WZ}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\})$ as shown in (15)–(16) on the next page as the sets of feasible transmit powers, compression noise levels, and receive beamforming vectors for the cases of independent compression and Wyner-Ziv compression under the decompression order of $\rho^{\text{ul}}(1), \dots, \rho^{\text{ul}}(M)$ at the CP, respectively. Then, for the considered multiple-access relay channel, the achievable rate regions for Case I: linear decoding at the CP and independent compression across the relays, Case II: successive interference cancellation at the CP and independent compression across the relays, Case III: linear decoding at the CP and Wyner-Ziv compression across the relays, and Case IV: successive interference cancellation at the CP and Wyner-Ziv compression across the

$$\mathcal{T}^{\text{ul,IN}}(\{C_m\}, P) = \{(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) : P^{\text{ul}}(\{p_k^{\text{ul}}\}) \leq P, C_m^{\text{ul}}(\{p_k^{\text{ul}}\}, q_m^{\text{ul}}) \leq C_m, q_m^{\text{ul}} \geq 0, \forall m \in \mathcal{M}, p_k^{\text{ul}} \geq 0, \|\mathbf{w}_k\|^2 = 1, \forall k \in \mathcal{K}\}, \quad (15)$$

$$\mathcal{T}^{\text{ul,WZ}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}) = \{(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) : P^{\text{ul}}(\{p_k^{\text{ul}}\}) \leq P, C_{\rho^{\text{ul}}(m)}^{\text{ul,WZ}}(\{p_k^{\text{ul}}\}, q_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, q_{\rho^{\text{ul}}(m)}^{\text{ul}}, \{p_k^{\text{ul}}(j)\}) \leq C_{\rho^{\text{ul}}(m)}, q_m^{\text{ul}} \geq 0, \forall m \in \mathcal{M}, p_k^{\text{ul}} \geq 0, \|\mathbf{w}_k\|^2 = 1, \forall k \in \mathcal{K}\}. \quad (16)$$

$$\bar{\mathcal{R}}_{\text{I}}^{\text{ul}}(\{C_m\}, P) \triangleq \bigcup_{(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) \in \mathcal{T}^{\text{ul,IN}}(\{C_m\}, P)} \left\{ (r_1^{\text{ul}}, \dots, r_K^{\text{ul}}) : r_k^{\text{ul}} \leq R_k^{\text{ul,TIN}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}), \forall k \in \mathcal{K} \right\}, \quad (21)$$

$$\bar{\mathcal{R}}_{\text{II}}^{\text{ul}}(\{C_m\}, P, \{\tau^{\text{ul}}(k)\}) \triangleq \bigcup_{(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) \in \mathcal{T}^{\text{ul,IN}}(\{C_m\}, P)} \left\{ (r_1^{\text{ul}}, \dots, r_K^{\text{ul}}) : r_{\tau^{\text{ul}}(k)}^{\text{ul}} \leq R_{\tau^{\text{ul}}(k)}^{\text{ul,SIC}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}, \{\tau^{\text{ul}}(k)\}), \forall k \in \mathcal{K} \right\}, \quad (22)$$

$$\bar{\mathcal{R}}_{\text{III}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}) \triangleq \bigcup_{(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) \in \mathcal{T}^{\text{ul,WZ}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\})} \left\{ (r_1^{\text{ul}}, \dots, r_K^{\text{ul}}) : r_k^{\text{ul}} \leq R_k^{\text{ul,TIN}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}), \forall k \in \mathcal{K} \right\}, \quad (23)$$

$$\bar{\mathcal{R}}_{\text{IV}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}, \{\tau^{\text{ul}}(k)\}) \triangleq \bigcup_{(\{p_k^{\text{ul}}, \mathbf{w}_i\}, \{q_m^{\text{ul}}\}) \in \mathcal{T}^{\text{ul,WZ}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\})} \left\{ (r_1^{\text{ul}}, \dots, r_K^{\text{ul}}) : r_{\tau^{\text{ul}}(k)}^{\text{ul}} \leq R_{\tau^{\text{ul}}(k)}^{\text{ul,SIC}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}, \{\tau^{\text{ul}}(k)\}), \forall k \in \mathcal{K} \right\}. \quad (24)$$

relays, are respectively given by [17], [47]

$$\mathcal{R}_{\text{I}}^{\text{ul}}(\{C_m\}, P) \triangleq \text{co } \bar{\mathcal{R}}_{\text{I}}^{\text{ul}}(\{C_m\}, P), \quad (17)$$

$$\mathcal{R}_{\text{II}}^{\text{ul}}(\{C_m\}, P) \triangleq \text{co } \bigcup_{\{\tau^{\text{ul}}(k)\}} \bar{\mathcal{R}}_{\text{II}}^{\text{ul}}(\{C_m\}, P, \{\tau^{\text{ul}}(k)\}), \quad (18)$$

$$\mathcal{R}_{\text{III}}^{\text{ul}}(\{C_m\}, P) \triangleq \text{co } \bigcup_{\{\rho^{\text{ul}}(m)\}} \bar{\mathcal{R}}_{\text{III}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}), \quad (19)$$

$$\mathcal{R}_{\text{IV}}^{\text{ul}}(\{C_m\}, P) \triangleq \text{co } \bigcup_{(\{\rho^{\text{ul}}(m)\}, \{\tau^{\text{ul}}(k)\})} \bar{\mathcal{R}}_{\text{IV}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}, \{\tau^{\text{ul}}(k)\}), \quad (20)$$

where ‘‘co’’ stands for the closure of convex hull operation and in (17), (18), (19), and (20), $\bar{\mathcal{R}}_{\text{I}}^{\text{ul}}(\{C_m\}, P)$, $\bar{\mathcal{R}}_{\text{II}}^{\text{ul}}(\{C_m\}, P, \{\tau^{\text{ul}}(k)\})$, $\bar{\mathcal{R}}_{\text{III}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\})$, and $\bar{\mathcal{R}}_{\text{IV}}^{\text{ul}}(\{C_m\}, P, \{\rho^{\text{ul}}(m)\}, \{\tau^{\text{ul}}(k)\})$ are expressed in (21)–(24) on top of the page and denote the rate regions of Case I, and Case II given the decoding order $\tau^{\text{ul}}(1), \dots, \tau^{\text{ul}}(K)$, and Case III given the decompression order $\rho^{\text{ul}}(1), \dots, \rho^{\text{ul}}(M)$, and Case IV given the decoding order $\tau^{\text{ul}}(1), \dots, \tau^{\text{ul}}(K)$ and the decompression order $\rho^{\text{ul}}(1), \dots, \rho^{\text{ul}}(M)$, respectively.

As a remark, because the achievable rates under the proposed transmission and relaying strategies are not necessarily concave functions of C_m and P , there is the potential to

further enlarge the above rate region by taking the convex hull over different C_m 's and P 's. For ease of presentation, the statements of the main results in this paper do not include this additional convex hull operation, but such an operation can be easily incorporated.

B. Broadcast Channel with Compressing Relays

The Gaussian broadcast relay channel model is as shown in Fig. 2(b). The discrete-time baseband channel model between the relays and the users is the dual of the broadcast relay channel given by

$$\begin{bmatrix} y_1^{\text{dl}} \\ \vdots \\ y_K^{\text{dl}} \end{bmatrix} = \begin{bmatrix} h_{1,1}^H & \cdots & h_{M,1}^H \\ \vdots & \ddots & \vdots \\ h_{1,K}^H & \cdots & h_{M,K}^H \end{bmatrix} \begin{bmatrix} x_1^{\text{dl}} \\ \vdots \\ x_M^{\text{dl}} \end{bmatrix} + \begin{bmatrix} z_1^{\text{dl}} \\ \vdots \\ z_K^{\text{dl}} \end{bmatrix}, \quad (25)$$

where x_m^{dl} denotes the transmit signal of relay m , and $z_k^{\text{dl}} \sim \mathcal{CN}(0, \sigma^2)$ denotes the AWGN at user k .

Transmission and relaying strategies for the broadcast relay channel have also been studied extensively in the literature. For example, the CP can choose to partially share the messages of each user with multiple BSs in order to enable cooperation [48]. This paper however focuses on a compression strategy in which the beamformed signals are precomputed at the CP,

$$\begin{aligned}
C_{\rho^{\text{dl}}(m)}^{\text{dl,MV}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\rho^{\text{dl}}(m)\}) &= I(x_{\rho^{\text{dl}}(m)}^{\text{dl}}; \tilde{x}_{\rho^{\text{dl}}(m)}^{\text{dl}} | \tilde{x}_{\rho^{\text{dl}}(1)}^{\text{dl}}, \dots, \tilde{x}_{\rho^{\text{dl}}(m-1)}^{\text{dl}}) \\
&= \log_2 \frac{\sum_{k=1}^K p_k^{\text{dl}} |v_{k, \rho^{\text{dl}}(m)}|^2 + \mathbf{Q}_{\{\rho^{\text{dl}}(m)\}}^{(m,m)}}{\mathbf{Q}_{\{\rho^{\text{dl}}(m)\}}^{(m,m)} - \mathbf{Q}_{\{\rho^{\text{dl}}(m)\}}^{(m,m+1:M)} (\mathbf{Q}_{\{\rho^{\text{dl}}(m)\}}^{(m+1:M,m+1:M)})^{-1} \mathbf{Q}_{\{\rho^{\text{dl}}(m)\}}^{(m+1:M,m)}}, \quad \forall m \in \mathcal{M}, \quad (33)
\end{aligned}$$

where

$$\mathbf{Q}_{\{\rho^{\text{dl}}(m)\}} = \mathbb{E} \left[[e_{\rho^{\text{dl}}(1)}^{\text{dl}}, \dots, e_{\rho^{\text{dl}}(M)}^{\text{dl}}]^T [(e_{\rho^{\text{dl}}(1)}^{\text{dl}})^*, \dots, (e_{\rho^{\text{dl}}(M)}^{\text{dl}})^*] \right]. \quad (34)$$

then compressed and forwarded to the relays [21], because of its potential to achieve the capacity region to within a constant gap [45], [46].

More specifically, we use a Gaussian codebook for each user and define the beamformed signal intended for user k to be transmitted across all the relays as $\mathbf{v}_k \sqrt{p_k^{\text{dl}}} s_k^{\text{dl}}, \forall k$, where $s_k^{\text{dl}} \sim \mathcal{CN}(0, 1)$ denotes the message for user k , $p_k^{\text{dl}} \geq 0$ denotes the transmit power, and $\mathbf{v}_k = [v_{k,1}, \dots, v_{k,M}]^T$ with $\|\mathbf{v}_k\|^2 = 1$ denotes the transmit beamforming vector across the relays. The aggregate signal intended for all the relays is thus given by

$$\tilde{\mathbf{x}}^{\text{dl}} = [\tilde{x}_1^{\text{dl}}, \dots, \tilde{x}_M^{\text{dl}}]^T = \sum_{k=1}^K \mathbf{v}_k \sqrt{p_k^{\text{dl}}} s_k^{\text{dl}}, \quad (26)$$

which is compressed then sent to the relays via fronthaul links.

Similar to (5), the quantization noises are modelled as Gaussian random variables, i.e.,

$$x_m^{\text{dl}} = \tilde{x}_m^{\text{dl}} + e_m^{\text{dl}}, \quad \forall m \in \mathcal{M}, \quad (27)$$

where $e_m^{\text{dl}} \sim \mathcal{CN}(0, q_m^{\text{dl}})$ denotes the quantization noise at relay m , with q_m^{dl} denoting its variance. Putting all the quantization noises across all the relays together as $\mathbf{e}^{\text{dl}} = [e_1^{\text{dl}}, \dots, e_M^{\text{dl}}]^T$, we have the quantization noise covariance matrix

$$\mathbf{Q} = \mathbb{E} \left[\mathbf{e}^{\text{dl}} (\mathbf{e}^{\text{dl}})^H \right] \succeq \mathbf{0}. \quad (28)$$

The compressed versions of the beamformed signals are transmitted across the relays. According to (27), the transmit signal can be expressed as

$$\begin{bmatrix} x_1^{\text{dl}} \\ \vdots \\ x_M^{\text{dl}} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^K v_{k,1} \sqrt{p_k^{\text{dl}}} s_k^{\text{dl}} \\ \vdots \\ \sum_{k=1}^K v_{k,M} \sqrt{p_k^{\text{dl}}} s_k^{\text{dl}} \end{bmatrix} + \begin{bmatrix} e_1^{\text{dl}} \\ \vdots \\ e_M^{\text{dl}} \end{bmatrix}. \quad (29)$$

Under the above model, the transmit power of all the relays is expressed as

$$P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}) = \sum_{m=1}^M \mathbb{E}[|x_m^{\text{dl}}|^2] = \sum_{k=1}^K p_k^{\text{dl}} + \text{tr}(\mathbf{Q}). \quad (30)$$

1) *Compression Strategies:* We consider two compression strategies in our considered broadcast relay channel: the independent compression strategy and the multivariate compression strategy. If the compression is done independently for the signals across different relays, then the covariance matrix

of the compression noise given in (28) reduces to a diagonal matrix, i.e.,

$$\mathbf{Q} = \mathbf{Q}_{\text{diag}} \triangleq \text{diag}([q_1^{\text{dl}}, \dots, q_M^{\text{dl}}]). \quad (31)$$

As a result, the fronthaul rate for transmitting x_m^{dl} is expressed as

$$\begin{aligned}
C_m^{\text{dl,IN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) &= I(\tilde{x}_m^{\text{dl}}; x_m^{\text{dl}}) \\
&= \log_2 \frac{\sum_{k=1}^K p_k^{\text{dl}} |v_{k,m}|^2 + \mathbf{Q}_{\text{diag}}^{(m,m)}}{\mathbf{Q}_{\text{diag}}^{(m,m)}}, \quad \forall m \in \mathcal{M}. \quad (32)
\end{aligned}$$

Alternatively, the CP can also use the multivariate compression strategy. Given a compression order at the CP $\rho^{\text{dl}}(1), \dots, \rho^{\text{dl}}(M)$ in which the signal for relay $\rho^{\text{dl}}(1) \in \mathcal{M}$ is compressed first, the signal for relay $\rho^{\text{dl}}(2)$ is compressed second, etc., the compression rate for relay $\rho^{\text{dl}}(m)$ can be expressed as (33)–(34) on top of the page [21].

2) *Encoding Strategies:* We consider two encoding strategies at the CP: the linear encoding strategy and the nonlinear encoding strategy via dirty-paper coding. If the CP employs linear encoding, the achievable rate of user k can be expressed as

$$\begin{aligned}
R_k^{\text{dl,LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}) &= I(s_k^{\text{dl}}; y_k^{\text{dl}}) \\
&= \log_2 \frac{\sum_{i=1}^K p_i^{\text{dl}} |\mathbf{v}_i^H \mathbf{h}_k|^2 + \mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k + \sigma^2}{\sum_{j \neq k} p_j^{\text{dl}} |\mathbf{v}_j^H \mathbf{h}_k|^2 + \mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k + \sigma^2}, \quad \forall k \in \mathcal{K}. \quad (35)
\end{aligned}$$

If the dirty-paper coding strategy is used, given an encoding order $\tau^{\text{dl}}(1), \dots, \tau^{\text{dl}}(K)$ at the CP in which the message of user $\tau^{\text{dl}}(1) \in \mathcal{K}$ is decoded first, the message of user $\tau^{\text{dl}}(2) \in \mathcal{K}$ is decoded second, etc., then the achievable rate of user $\tau^{\text{dl}}(k)$ is expressed as

$$\begin{aligned}
R_{\tau^{\text{dl}}(k)}^{\text{dl,DPC}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\tau^{\text{dl}}(k)\}) &= I(s_{\tau^{\text{dl}}(k)}^{\text{dl}}; y_{\tau^{\text{dl}}(k)}^{\text{dl}} | s_{\tau^{\text{dl}}(1)}^{\text{dl}}, \dots, s_{\tau^{\text{dl}}(k-1)}^{\text{dl}}) \\
&= \log_2 \frac{\sum_{i \leq k} p_{\tau^{\text{dl}}(i)}^{\text{dl}} |\mathbf{v}_{\tau^{\text{dl}}(i)}^H \mathbf{h}_{\tau^{\text{dl}}(k)}|^2 + \mathbf{h}_{\tau^{\text{dl}}(k)}^H \mathbf{Q} \mathbf{h}_{\tau^{\text{dl}}(k)} + \sigma^2}{\sum_{j < k} p_{\tau^{\text{dl}}(j)}^{\text{dl}} |\mathbf{v}_{\tau^{\text{dl}}(j)}^H \mathbf{h}_{\tau^{\text{dl}}(k)}|^2 + \mathbf{h}_{\tau^{\text{dl}}(k)}^H \mathbf{Q} \mathbf{h}_{\tau^{\text{dl}}(k)} + \sigma^2}, \\
&\quad \forall k \in \mathcal{K}. \quad (36)
\end{aligned}$$

In (35) and (36), if we set $\mathbf{Q} = \mathbf{Q}_{\text{diag}}$ as shown in (31), then $R_k^{\text{dl,LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}})$'s and $R_{\tau^{\text{dl}}(k)}^{\text{dl,DPC}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}, \{\tau^{\text{dl}}(k)\})$'s denote the user rates

$$\mathcal{T}^{\text{dl,IN}}(\{C_m\}, P) = \{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) : P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}_{\text{diag}}) \leq P, \mathbf{Q}_{\text{diag}} \succeq \mathbf{0} \text{ is diagonal}, \\ C_m^{\text{dl,IN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) \leq C_m, \forall m \in \mathcal{M}, p_k^{\text{dl}} \geq 0, \|\mathbf{v}_k\|^2 = 1, \forall k \in \mathcal{K}\}, \quad (37)$$

$$\mathcal{T}^{\text{dl,MV}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}) = \{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}) : P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}) \leq P, \mathbf{Q} \succeq \mathbf{0}, \\ C_{\rho^{\text{dl}}(m)}^{\text{dl,MV}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\rho^{\text{dl}}(m)\}) \leq C_{\rho^{\text{dl}}(m)}, \forall m \in \mathcal{M}, p_k^{\text{dl}} \geq 0, \|\mathbf{v}_k\|^2 = 1, \forall k \in \mathcal{K}\}. \quad (38)$$

achieved by the independent compression strategy. If \mathbf{Q} is a full matrix (i.e., non-diagonal), then $R_k^{\text{dl,LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q})$'s and $R_{\tau^{\text{dl}}(k)}^{\text{dl,DPC}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\tau^{\text{dl}}(k)\})$'s denote the user rates achieved by the multivariate compression strategy.

3) *Achievable Rate Regions*: Given the individual fronthaul capacity constraints $\{C_m\}$ and sum-power constraint P , define $\mathcal{T}^{\text{dl,IN}}(\{C_m\}, P)$ and $\mathcal{T}^{\text{dl,MV}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\})$ as shown in (37)–(38) on top of the page as the sets of feasible transmit powers, beamforming vectors, and compression noise covariance matrices for the cases of independent compression and multivariate compression under the compression order of $\rho^{\text{dl}}(1), \dots, \rho^{\text{dl}}(M)$, respectively. Then, in the broadcast relay channel, the achievable rate regions for Case I: linear encoding and independent compression at the CP, Case II: dirty-paper coding and independent compression at the CP, Case III: linear encoding and multivariate compression at the CP, and Case IV: dirty-paper coding and multivariate compression at the CP, are respectively given by [21], [46]

$$\mathcal{R}_I^{\text{dl}}(\{C_m\}, P) \triangleq \text{co} \bar{\mathcal{R}}_I^{\text{dl}}(\{C_m\}, P) \quad (39)$$

$$\mathcal{R}_{\text{II}}^{\text{dl}}(\{C_m\}, P) \triangleq \text{co} \bigcup_{\{\tau^{\text{dl}}(k)\}} \bar{\mathcal{R}}_{\text{II}}^{\text{dl}}(\{C_m\}, P, \{\tau^{\text{dl}}(k)\}), \quad (40)$$

$$\mathcal{R}_{\text{III}}^{\text{dl}}(\{C_m\}, P) \triangleq \text{co} \bigcup_{\{\rho^{\text{dl}}(m)\}} \bar{\mathcal{R}}_{\text{III}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}), \quad (41)$$

$$\mathcal{R}_{\text{IV}}^{\text{dl}}(\{C_m\}, P) \triangleq \\ \text{co} \bigcup_{(\{\rho^{\text{dl}}(m)\}, \{\tau^{\text{dl}}(k)\})} \bar{\mathcal{R}}_{\text{IV}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}, \{\tau^{\text{dl}}(k)\}), \quad (42)$$

where the rate regions denoted by $\bar{\mathcal{R}}_I^{\text{dl}}(\{C_m\}, P)$, $\bar{\mathcal{R}}_{\text{II}}^{\text{dl}}(\{C_m\}, P, \{\tau^{\text{dl}}(k)\})$, $\bar{\mathcal{R}}_{\text{III}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\})$, and $\bar{\mathcal{R}}_{\text{IV}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}, \{\tau^{\text{dl}}(k)\})$ and expressed in (43)–(46) on the next page are the rate regions of Case I, and Case II given the encoding order $\tau^{\text{dl}}(1), \dots, \tau^{\text{dl}}(K)$, and Case III given the compression order $\rho^{\text{dl}}(1), \dots, \rho^{\text{dl}}(M)$, and Case IV given the encoding order $\tau^{\text{dl}}(1), \dots, \tau^{\text{dl}}(K)$ and the compression order $\rho^{\text{dl}}(1), \dots, \rho^{\text{dl}}(M)$, respectively.

As a remark, similar to the multiple-access relay channel case, an additional closure of convex hull operation can be applied over the C_m 's and P 's to potentially enlarge the achievable rate region. The statements of main results in this paper do not include this extra convex hull operation for simplicity, but it can be easily incorporated in all the statements of the theorems and the proofs.

III. MAIN RESULTS

The main results of this work are the following set of theorems showing the duality relationships between the achievable rate regions of the multiple-access relay channel and the broadcast relay channel under the same sum-power constraint and individual fronthaul constraints.

Theorem 1: Consider the multiple-access relay channel implementing independent compression across the relays as well as linear decoding at the CP and the broadcast relay channel implementing independent compression across the relays as well as linear encoding at the CP, where all the users and relays are equipped with a single antenna. Then, under the same sum-power constraint P and individual fronthaul capacity constraints C_m 's, the achievable rate regions of the multiple-access relay channel defined in (17) and the broadcast relay channel defined in (39) are identical. In other words,

$$\mathcal{R}_I^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_I^{\text{dl}}(\{C_m\}, P). \quad (47)$$

Theorem 2: Consider the multiple-access relay channel implementing independent compression across the relays as well as successive interference cancellation at the CP and the broadcast relay channel implementing independent compression across the relays as well as dirty-paper coding at the CP, where all the users and relays are equipped with a single antenna. Then, under the same sum-power constraint P and individual fronthaul capacity constraints C_m 's, the achievable rate regions of the multiple-access relay channel defined in (18) and the broadcast relay channel defined in (40) are identical. In other words,

$$\mathcal{R}_{\text{II}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{II}}^{\text{dl}}(\{C_m\}, P). \quad (48)$$

Theorem 3: Consider the multiple-access relay channel implementing Wyner-Ziv compression across the relays as well as linear decoding at the CP and the broadcast relay channel implementing multivariate compression across the relays as well as linear encoding at the CP, where all the users and relays are equipped with a single antenna. Then, under the same sum-power constraint P and individual fronthaul capacity constraints C_m 's, the achievable rate regions of the multiple-access relay channel defined in (19) and the broadcast relay channel defined in (41) are identical. In other words,

$$\mathcal{R}_{\text{III}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{III}}^{\text{dl}}(\{C_m\}, P). \quad (49)$$

Theorem 4: Consider the multiple-access relay channel implementing Wyner-Ziv compression across the relays as well as successive interference cancellation at the CP and the broadcast relay channel implementing multivariate compression across the relays as well as dirty-paper coding at

$$\bar{\mathcal{R}}_{\text{I}}^{\text{dl}}(\{C_m\}, P) \triangleq \bigcup_{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) \in \mathcal{T}^{\text{dl}, \text{LIN}}(\{C_m\}, P)} \left\{ (r_1^{\text{dl}}, \dots, r_K^{\text{dl}}) : \right. \\ \left. r_k^{\text{dl}} \leq R_k^{\text{dl}, \text{LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}), \forall k \in \mathcal{K} \right\}, \quad (43)$$

$$\bar{\mathcal{R}}_{\text{II}}^{\text{dl}}(\{C_m\}, P, \{\tau^{\text{dl}}(k)\}) \triangleq \bigcup_{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) \in \mathcal{T}^{\text{dl}, \text{LIN}}(\{C_m\}, P)} \left\{ (r_1^{\text{dl}}, \dots, r_K^{\text{dl}}) : \right. \\ \left. r_{\tau^{\text{dl}}(k)}^{\text{dl}} \leq R_{\tau^{\text{dl}}(k)}^{\text{dl}, \text{DPC}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}, \{\tau^{\text{dl}}(k)\}), \forall k \in \mathcal{K} \right\}, \quad (44)$$

$$\bar{\mathcal{R}}_{\text{III}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}) \triangleq \bigcup_{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}) \in \mathcal{T}^{\text{dl}, \text{MV}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\})} \left\{ (r_1^{\text{dl}}, \dots, r_K^{\text{dl}}) : \right. \\ \left. r_k^{\text{dl}} \leq R_k^{\text{dl}, \text{LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}), \forall k \in \mathcal{K} \right\}, \quad (45)$$

$$\bar{\mathcal{R}}_{\text{IV}}^{\text{dl}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\}, \{\tau^{\text{dl}}(k)\}) \triangleq \bigcup_{(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}) \in \mathcal{T}^{\text{dl}, \text{MV}}(\{C_m\}, P, \{\rho^{\text{dl}}(m)\})} \left\{ (r_1^{\text{dl}}, \dots, r_K^{\text{dl}}) : \right. \\ \left. r_{\tau^{\text{dl}}(k)}^{\text{dl}} \leq R_{\tau^{\text{dl}}(k)}^{\text{dl}, \text{DPC}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\tau^{\text{dl}}(k)\}), \forall k \in \mathcal{K} \right\}. \quad (46)$$

the CP, where all the users and relays are equipped with a single antenna. Then, under the same sum-power constraint P and individual fronthaul capacity constraints C_m 's, the achievable rate regions of the multiple-access relay channel defined in (20) and the broadcast relay channel defined in (42) are identical. In other words,

$$\mathcal{R}_{\text{IV}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{IV}}^{\text{dl}}(\{C_m\}, P). \quad (50)$$

As mentioned earlier, when the fronthaul capacity of each relay is infinite, i.e., $C_m \rightarrow \infty, \forall m$, the quantization noises can be set to zero. As a result, the M relays and the CP are equivalent to a virtual BS with M antennas. In this case, the multiple-access relay channel and the broadcast relay channel reduce to the usual multiple-access channel and broadcast channel, respectively; therefore, the classic uplink-downlink duality directly applies. Our main results, i.e., Theorems 1 to 4, are generalizations of the classic uplink-downlink duality result to the case with non-zero quantization noises.

We note that the exact capacity region characterizations of the multiple-access relay channel and the broadcast relay channel are both still open problems. The duality results above pertain to the specific compression-based relaying strategies. Although it is possible to outperform these strategies for specific channel instances, the compression strategies are important both in practical implementations [4] and due to its ability to approximately approach the theoretical capacity regions under specific conditions for both the uplink [17] and the downlink [46] as mentioned earlier.

It is worth noting that as shown in [1], [11], with dirty paper coding in the broadcast channel and successive interference cancellation in the multiple-access channel (without relays), to achieve the same rate tuple with the same sum power in the downlink and the uplink, the encoding order and the decoding order should be reverse of each other. In this paper, we show a

similar result for the broadcast channel and the multiple-access channel with compressing relays: if multivariate compression and Wyner-Ziv compression are used by the relays, then to achieve the same rate tuple with the same sum power and the same fronthaul capacities in the downlink and the uplink, the compression order and the decompression order should also be reverse of each other. Please note the clear analogy between dirty paper coding and multivariate compression (in the sense that they both cancel the interference caused by encoding/compression seen by the users) and successive interference cancellation and Wyner-Ziv compression (in the sense that the decoded/decompressed signals can be used for decoding/decompressing the remaining signals). Rigorous proofs of these results are provided in Sections VI and VII.

IV. PROOF OF THEOREM 1

In this section, we prove the duality between the multiple-access relay channel with linear decoding at the CP as well as independent compression across the relays and the broadcast relay channel with linear encoding at the CP and independent compression across the relays. Suppose that the same beamforming vectors

$$\mathbf{v}_k = \mathbf{w}_k = \bar{\mathbf{u}}_k = [\bar{u}_{k,1}, \dots, \bar{u}_{k,M}]^T, \quad \forall k \in \mathcal{K}, \quad (51)$$

with $\|\bar{\mathbf{u}}_k\| = 1, \forall k \in \mathcal{K}$, are used in both the multiple-access relay channel and the broadcast relay channel. For simplicity, we assume that the beamforming vectors $\bar{\mathbf{u}}_k$'s satisfy the following condition:

$$\sum_{k=1}^K |\bar{u}_{k,m}|^2 > 0, \quad \forall m \in \mathcal{M}. \quad (52)$$

Condition (52) indicates that all the relays are used for communications so that the fronthaul rates are properly defined.

If we have $\sum_{k=1}^K |\bar{u}_{k,m}|^2 = 0$ for some m , then we can simply define $\mathcal{M} = \{m : \sum_{k=1}^K |\bar{u}_{k,m}|^2 > 0\}$ and $\bar{M} = |\bar{\mathcal{M}}|$, respectively. In this case, the considered system is equivalent to a system merely consisting of \bar{M} relays in the set of $\bar{\mathcal{M}}$, in which we have $\sum_{k=1}^K |\bar{u}_{k,m}|^2 > 0, \forall m \in \bar{\mathcal{M}}$. As a result, condition (52) does not affect the generality of our following results.

Let $\{R_k \geq 0, k \in \mathcal{K}\}$ be a set of user target rates and $\{C_m \geq 0, m \in \mathcal{M}\}$ be a set of fronthaul rate requirements for the relays. For the multiple-access relay channel as described in Section II-A, we fix the receive beamforming vectors as in (51) and formulate the transmit power minimization problem subject to the individual rate constraints as well as the individual fronthaul capacity constraints as follows:

$$\text{minimize } P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (53)$$

$$\text{subject to } R_k^{\text{ul},\text{TIN}}(\{p_k^{\text{ul}}, \bar{\mathbf{u}}_k\}, \{q_m^{\text{ul}}\}) \geq R_k, \forall k \in \mathcal{K}, \quad (54)$$

$$C_m^{\text{ul},\text{IN}}(\{p_k^{\text{ul}}\}, q_m^{\text{ul}}) \leq C_m, \forall m \in \mathcal{M}, \quad (55)$$

$$p_k^{\text{ul}} \geq 0, \forall k \in \mathcal{K}, \quad (56)$$

$$q_m^{\text{ul}} \geq 0, \forall m \in \mathcal{M}. \quad (57)$$

Similarly, for the broadcast relay channel as described in Section II-B, we fix the transmit beamforming vectors as in (51) and formulate the transmit power minimization problem as

$$\text{minimize } P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}_{\text{diag}}) \quad (58)$$

$$\text{subject to } R_k^{\text{dl},\text{LIN}}(\{p_k^{\text{dl}}, \bar{\mathbf{u}}_k\}, \mathbf{Q}_{\text{diag}}) \geq R_k, \forall k \in \mathcal{K}, \quad (59)$$

$$C_m^{\text{dl},\text{IN}}(\{p_k^{\text{dl}}, \bar{\mathbf{u}}_k\}, \mathbf{Q}_{\text{diag}}) \leq C_m, \forall m \in \mathcal{M}, \quad (60)$$

$$p_k^{\text{dl}} \geq 0, \forall k \in \mathcal{K}, \quad (61)$$

$$\mathbf{Q}_{\text{diag}}^{(m,m)} \geq 0, \forall m \in \mathcal{M}. \quad (62)$$

Our aim is to show that given the same fixed beamformer and under the same set of rate targets $\{R_k\}$, the optimization problems (53) and (58) are equivalent in the sense that either both are infeasible, or both are feasible and have the same minimum solution. This would imply that under the same fronthaul capacity constraints $\{C_m\}$ and total transmit power constraint P , fixing the same beamformers, any achievable rate-tuple $\{R_k\}$ of the multiple-access relay channel is also achievable in the broadcast relay channel, and vice versa. Then, by trying all beamforming vectors, this would imply that the achievable rate regions of the multiple-access relay channel under linear decoding as well as independent compression and the broadcast relay channel under linear encoding as well as independent compression are identical, i.e., $\bar{\mathcal{R}}_1^{\text{ul}}(\{C_m\}, P) = \bar{\mathcal{R}}_1^{\text{dl}}(\{C_m\}, P)$. Finally, by taking convex hull, we get $\mathcal{R}_1^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_1^{\text{dl}}(\{C_m\}, P)$.

To show the equivalence of the optimization problems (53) and (58) for the fixed beamformers, we take a set of $\{R_k\}$ and $\{C_m\}$ such that both (53) and (58) are strictly feasible, and show that (53) and (58) can both be transformed into convex optimization problems. Further, we show that the two convex formulations are the Lagrangian duals of each other, which implies that they must have the same minimum sum

power. Once this is proved, we can further infer that the feasible rate regions of the two problems are identical. This is because the feasible rate regions can be equivalently viewed as the sets of rate-tuples for which the minimum values of the optimization problems (53) and (58) are less than infinity. As both (53) and (58) can be reformulated as convex problems, the minimum powers in the two problems are convex functions of $\{R_k\}$ and $\{C_m\}$ [41]. This means that the minimum powers in the two problems are continuous functions of $\{R_k\}$. Since the minimum powers of the two problems are the same whenever $\{R_k\}$ is strictly feasible for both problems, as $\{R_k\}$ approaches the feasibility boundary of one problem, the minimum powers for both problems must go to infinity at the same time, implying that the same $\{R_k\}$ must be approaching the feasibility boundary of the other problem as well; thus the two problems must have the same feasibility region. This same argument also applies to the proofs of Theorems 2 to 4. Thus in the rest of the proofs of all four theorems, we only show that given a set of user rates $\{R_k\}$ and fronthaul capacities $\{C_m\}$ that are strictly feasible in both the multiple-access relay channel and the broadcast relay channel under the fixed beamforming vectors (51), the minimum sum powers of the two problems are the same.

We remark that our previous work [40] provides a different approach to validate the equivalence between (53) and (58) based on the classic power control technique. This paper uses the alternative approach of showing the equivalence between (53) and (58) based on a Lagrangian duality technique. This allows a unified approach for proving Theorems 1 to 4. The proof involves the following two steps.

A. Convex Reformulation of Problem (58) and Its Dual Problem

First, we transform the problem (58) for the broadcast relay channel into the following convex optimization problem:

$$\text{minimize } \sum_{k=1}^K p_k^{\text{dl}} \sigma^2 + \sum_{m=1}^M q_m^{\text{dl}} \sigma^2 \quad (63)$$

$$\text{subject to } \frac{p_k^{\text{dl}} |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} - \sum_{j \neq k} p_j^{\text{dl}} |\bar{\mathbf{u}}_j^H \mathbf{h}_k|^2 - \sum_{m=1}^M q_m^{\text{dl}} |h_{k,m}|^2 - \sigma^2 \geq 0, \forall k \in \mathcal{K}, \quad (64)$$

$$\sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,m}|^2 + q_m^{\text{dl}} - 2^{C_m} q_m^{\text{dl}} \leq 0, \forall m \in \mathcal{M}, \quad (65)$$

$$(61), (62).$$

Note that in the objective function, the sum power is multiplied by a constant σ^2 without loss of generality. In fact, problem (63) is an LP, since the objective function and constraints are all linear in p_k^{dl} 's and q_m^{dl} 's.

Take a set of strictly feasible $\{R_k\}$. Since the problem (63) is a convex problem, strong duality holds [41], i.e., problem (63) is equivalent to its dual problem. In the following, we derive the dual problem of the problem (63).

The Lagrangian of the problem (63) is (66) shown at the top of the next page, where $\beta_k \geq 0$'s and $\lambda_m \geq 0$'s are the dual

$$\begin{aligned}
L(\{p_k^{\text{dl}}, \beta_k\}, \{q_m^{\text{dl}}, \lambda_m\}) &= \sum_{k=1}^K p_k^{\text{dl}} \sigma^2 + \sum_{m=1}^M q_m^{\text{dl}} \sigma^2 - \sum_{k=1}^K \beta_k \left(\frac{p_k^{\text{dl}} |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} - \sum_{j \neq k} p_j^{\text{dl}} |\bar{\mathbf{u}}_j^H \mathbf{h}_k|^2 - \sum_{m=1}^M q_m^{\text{dl}} |h_{k,m}|^2 - \sigma^2 \right) \\
&\quad + \sum_{m=1}^M \lambda_m \left(\sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,m}|^2 + q_m^{\text{dl}} - 2^{C_m} q_m^{\text{dl}} \right) \\
&= \sum_{k=1}^K \beta_k \sigma^2 + \sum_{k=1}^K p_k^{\text{dl}} \left(\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m |\bar{u}_{k,m}|^2 + \sigma^2 - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \right) \\
&\quad + \sum_{m=1}^M q_m^{\text{dl}} \left(\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \lambda_m + \sigma^2 - 2^{C_m} \lambda_m \right). \tag{66}
\end{aligned}$$

variables associated with constraints (64) and (65) in problem (63), respectively. The dual function is then defined as

$$\begin{aligned}
g(\{\beta_k\}, \{\lambda_m\}) \\
= \min_{p_k^{\text{dl}} \geq 0, \forall k \in \mathcal{K}, q_m^{\text{dl}} \geq 0, \forall m \in \mathcal{M}} L(\{p_k^{\text{dl}}, \beta_k\}, \{q_m^{\text{dl}}, \lambda_m\}) \tag{67}
\end{aligned}$$

Finally, the dual problem of problem (63) is expressed as

$$\begin{aligned}
\text{maximize } g(\{\beta_k\}, \{\lambda_m\}) \tag{68} \\
\text{subject to } \beta_k \geq 0, \quad \forall k \in \mathcal{K}, \tag{69} \\
\lambda_m \geq 0, \quad \forall m \in \mathcal{M}. \tag{70}
\end{aligned}$$

Note that according to (67), $g(\{\beta_k\}, \{\lambda_m\}) = \sum_{k=1}^K \beta_k \sigma^2$ if and only if

$$\begin{aligned}
\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m |\bar{u}_{k,m}|^2 + \sigma^2 \\
- \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \geq 0, \quad \forall k \in \mathcal{K}, \tag{71}
\end{aligned}$$

$$\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \lambda_m + \sigma^2 - 2^{C_m} \lambda_m \geq 0, \quad \forall m \in \mathcal{M}. \tag{72}$$

Otherwise, $g(\{\beta_k\}, \{\lambda_m\}) = -\infty$. As a result, problem (68) can be transformed into the following equivalent problem:

$$\begin{aligned}
\text{maximize } \sum_{k=1}^K \beta_k \sigma^2 \tag{73} \\
\text{subject to (69), (70), (71), (72)}.
\end{aligned}$$

This problem is now very similar to the multiple-access relay channel problem. The physical interpretation of the above dual problem is the following. We can view β_k as the transmit power of user k , $\forall k \in \mathcal{K}$, and λ_m as the quantization noise level of relay m in the multiple-access relay channel, $\forall m \in \mathcal{M}$. Then, problem (73) aims to maximize the sum-power of all the users, while constraint (71) requires that user k 's rate is no larger than R_k , $\forall k \in \mathcal{K}$, and constraint (72) requires that relay m 's fronthaul rate is no smaller than C_m , $\forall m \in \mathcal{M}$. In the following, we show that for the multiple-access relay channel, this power maximization problem (73) is equivalent to the power minimization problem (53).

B. Equivalence Between Power Maximization Problem (73) and Power Minimization Problem (53)

To show the equivalence between problem (73) and problem (53), we first have the following proposition.

Proposition 1: At the optimal solution to problem (73), constraints (71) and (72) should hold with equality, i.e.,

$$\begin{aligned}
\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m |\bar{u}_{k,m}|^2 + \sigma^2 \\
- \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} = 0, \quad \forall k \in \mathcal{K}, \tag{74}
\end{aligned}$$

$$\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \lambda_m + \sigma^2 - 2^{C_m} \lambda_m = 0, \quad \forall m \in \mathcal{M}. \tag{75}$$

Proof: Please refer to Appendix A. ■

To satisfy (69), (70), (74), and (75), it can be shown that

$$\beta_k > 0, \quad \forall k \in \mathcal{K}, \tag{76}$$

$$\lambda_m > 0, \quad \forall m \in \mathcal{M}. \tag{77}$$

As a result, Proposition 1 indicates that problem (73) is equivalent to the following problem

$$\begin{aligned}
\text{maximize } \sum_{k=1}^K \beta_k \sigma^2 \tag{78} \\
\text{subject to (74), (75), (76), (77)}.
\end{aligned}$$

Proposition 2: If there exists one set of solutions β_k 's and λ_m 's that satisfies (74), (75), (76), and (77) in problem (78), then this solution is unique.

Proof: Please refer to Appendix B. ■

Since there is a unique solution that satisfies all the constraints in problem (78), it follows that the maximization problem (73) is equivalent to the following minimization problem:

$$\begin{aligned}
\text{minimize } \sum_{k=1}^K \beta_k \sigma^2 \tag{79} \\
\text{subject to (74), (75), (76), (77)}.
\end{aligned}$$

At last, we relate problem (79) to the power minimization problem (53) in the multiple-access relay channel by the following proposition.

Proposition 3: Problem (79) is equivalent to

$$\begin{aligned} & \underset{\{\beta_k\}, \{\lambda_m\}}{\text{minimize}} \sum_{k=1}^K \beta_k \sigma^2 & (80) \\ & \text{subject to} \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m |\bar{u}_{k,m}|^2 + \sigma^2 \\ & \quad - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2R_k - 1} \leq 0, \quad \forall k \in \mathcal{K}, & (81) \\ & \quad \sum_{k=1}^K \beta_k |h_{k,m}|^2 + \lambda_m + \sigma^2 - 2^{C_m} \lambda_m \leq 0, \\ & \quad \forall m \in \mathcal{M}, & (82) \end{aligned}$$

(76), (77).

Proof: Similar to Proposition 1, it can be shown that the optimal solution to problem (80) must have its constraints (81) and (82) satisfied with equality. As a result, problems (79) and (80) are equivalent to each other. ■

The equivalence between the dual problem of the problem (63), i.e., problem (73), and the problem (80) is therefore established. The key point here is that by viewing the dual variable β_k as the transmit power of user k , $\forall k \in \mathcal{K}$, and the dual variable λ_m as the quantization noise level of relay m , $\forall m \in \mathcal{M}$, in the multiple-access relay channel, the problem (80) is exactly the power minimization problem (53). As a result, we have shown that the problem (58) for the broadcast relay channel is equivalent to the problem (53) for the multiple-access relay channel if the user rate targets $\{R_k\}$ and fronthaul rate constraints $\{C_m\}$ are strictly feasible in both the uplink and downlink.

V. PROOF OF THEOREM 2

In this section, we prove the duality between the multiple-access relay channel with successive interference cancellation at the CP as well as independent compression across the relays and the broadcast relay channel with dirty-paper coding at the CP as well as independent compression across the relays.

Similar to Section IV, we fix the same beamforming vectors $\bar{\mathbf{u}}_k$'s in the multiple-access relay channel and the broadcast relay channel as shown in (51), where $\bar{\mathbf{u}}_k$'s satisfy (52). Next, we assume that the successive interference cancellation order at the CP for the multiple-access relay channel and the dirty-paper encoding order at the CP for the broadcast relay channel are reverse of each other, i.e.,

$$\tau^{\text{ul}}(k) = \tau^{\text{dl}}(K + 1 - k) = \bar{\tau}(k), \quad \forall k \in \mathcal{K}. \quad (83)$$

For example, if the decoding order in the multiple-access relay channel is $1, \dots, K$, then the encoding order in the broadcast relay channel is $K, \dots, 1$, i.e., $\bar{\tau}^{\text{ul}}(k) = k, \forall k$.

Let $\{R_k > 0, k \in \mathcal{K}\}$ and $\{C_m > 0, m \in \mathcal{M}\}$ be sets of strictly feasible user rate targets and fronthaul constraints for both the uplink and the downlink. Given the receive beamforming vectors (51) and decoding order (83), for the multiple-access relay channel, the transmit power minimization problem subject to the individual rate constraints as well

as the individual fronthaul capacity constraints is formulated as

$$\begin{aligned} & \underset{\{p_k^{\text{ul}}\}, \{q_m^{\text{ul}}\}}{\text{minimize}} P^{\text{ul}}(\{p_k^{\text{ul}}\}) & (84) \\ & \text{subject to} R_{\bar{\tau}(k)}^{\text{ul}, \text{SIC}}(\{p_k^{\text{ul}}, \bar{\mathbf{u}}_k\}, \{q_m^{\text{ul}}\}, \{\bar{\tau}(k)\}) \\ & \quad \geq R_{\bar{\tau}(k)}, \quad \forall k \in \mathcal{K}, & (85) \end{aligned}$$

(55), (56), (57).

Likewise, given the transmit beamforming vectors (51) and a reverse encoding order (83), for the broadcast relay channel, the transmit power minimization problem is formulated as

$$\begin{aligned} & \underset{\{p_k^{\text{dl}}\}, \mathbf{Q}_{\text{diag}}}{\text{minimize}} P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}_{\text{diag}}) & (86) \\ & \text{subject to} R_{\bar{\tau}(K+1-k)}^{\text{dl}, \text{DPC}}(\{p_k^{\text{dl}}, \bar{\mathbf{u}}_k\}, \mathbf{Q}_{\text{diag}}, \{\bar{\tau}(K+1-k)\}) \\ & \quad \geq R_{\bar{\tau}(K+1-k)}, \quad \forall k \in \mathcal{K}, & (87) \end{aligned}$$

(60), (61), (62).

Similar to the equivalence between problem (53) and problem (73) in Case I, it can be shown that problem (84) is equivalent to the following convex problem:

$$\begin{aligned} & \underset{\{p_k^{\text{ul}}\}, \{q_m^{\text{ul}}\}}{\text{maximize}} \sum_{k=1}^K p_k^{\text{ul}} \sigma^2 & (88) \\ & \text{subject to} \sum_{j>k} p_{\bar{\tau}(j)}^{\text{ul}} |\bar{\mathbf{u}}_{\bar{\tau}(k)}^H \mathbf{h}_{\bar{\tau}(j)}|^2 + \sum_{m=1}^M q_m^{\text{ul}} |\bar{u}_{\bar{\tau}(k),m}|^2 + \sigma^2 \\ & \quad \geq \frac{p_{\bar{\tau}(k)}^{\text{ul}} |\bar{\mathbf{u}}_{\bar{\tau}(k)}^H \mathbf{h}_{\bar{\tau}(k)}|^2}{2R_{\bar{\tau}(k)} - 1}, \quad \forall k \in \mathcal{K}, & (89) \\ & \quad \sum_{k=1}^K p_k^{\text{ul}} |h_{k,m}|^2 + q_m^{\text{ul}} + \sigma^2 \geq 2^{C_m} q_m^{\text{ul}}, \quad \forall m \in \mathcal{M}, & (90) \end{aligned}$$

(56), (57).

Moreover, we can transform problem (86) into the following convex problem:

$$\begin{aligned} & \underset{\{p_k^{\text{dl}}\}, \{q_m^{\text{dl}}\}}{\text{minimize}} \sum_{k=1}^K p_k^{\text{dl}} \sigma^2 + \sum_{m=1}^M q_m^{\text{dl}} \sigma^2 & (91) \\ & \text{subject to} \frac{p_{\bar{\tau}(k)}^{\text{dl}} |\bar{\mathbf{u}}_{\bar{\tau}(k)}^H \mathbf{h}_{\bar{\tau}(k)}|^2}{2R_{\bar{\tau}(k)} - 1} \geq \sum_{j \leq k} p_{\bar{\tau}(j)}^{\text{dl}} |\mathbf{v}_{\bar{\tau}(j)}^H \mathbf{h}_{\bar{\tau}(k)}|^2 \\ & \quad + \sum_{m=1}^M q_m^{\text{dl}} |h_{\bar{\tau}(k),m}|^2 + \sigma^2, \quad \forall k \in \mathcal{K}, & (92) \end{aligned}$$

(65), (61), (62).

Similar to Case I, we can show that problem (88) is the Lagrangian dual of problem (91). Since the method adopted is almost exactly the same as that in Section IV, we omit the details here. As a result, under the same fronthaul capacity constraints $\{C_m\}$ and total transmit power constraint P , any rate-tuple achievable in the multiple-access relay channel can be shown to be achievable also in the broadcast relay channel by setting the transmit beamforming vectors as the receive beamforming vectors in the multiple-access relay channel and the encoding order to be reverse of the decoding order in

the multiple-access channel, and vice versa. By trying all the feasible beamforming vectors, it follows that

$$\bar{\mathcal{R}}_{\text{II}}^{\text{ul}}(\{C_m\}, P, \{\bar{\tau}(k)\}) = \bar{\mathcal{R}}_{\text{II}}^{\text{dl}}(\{C_m\}, P, \{\bar{\tau}(K+1-k)\}). \quad (93)$$

Then, by trying all the encoding/decoding orders, we can show that the achievable rate regions of the multiple-access relay channel under successive interference cancellation as well as independent compression and the broadcast relay channel under dirty-paper coding as well as independent compression are identical, i.e., $\mathcal{R}_{\text{II}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{II}}^{\text{dl}}(\{C_m\}, P)$.

VI. PROOF OF THEOREM 3

In this section, we prove the duality between the multiple-access relay channel with linear decoding at the CP as well as Wyner-Ziv compression across the relays and the broadcast relay channel with linear encoding at the CP as well as multivariate compression across the relays.

Similar to Sections IV and V, we fix the same beamforming vectors $\bar{\mathbf{u}}_k$'s in the multiple-access relay channel and the broadcast relay channel as shown in (51), where $\bar{\mathbf{u}}_k$'s satisfy (52). Next, for Wyner-Ziv compression across the relays in the multiple-access relay channel and multivariate compression across relays in the broadcast relay channel, we assume that the decompression order is the reverse of the compression order, i.e.,

$$\rho^{\text{ul}}(m) = \rho^{\text{dl}}(M+1-m) = \bar{\rho}(m), \quad \forall m \in \mathcal{M}. \quad (94)$$

For example, if the decompression order in the multiple-access relay channel is $1, \dots, M$, then the compression order in the broadcast relay channel is $M, \dots, 1$, i.e., $\bar{\rho}(m) = m, \forall m$.

Let $\{R_k > 0, k \in \mathcal{K}\}$ and $\{C_m > 0, m \in \mathcal{M}\}$ be sets of strictly feasible user rate targets and fronthaul constraints for both the uplink and the downlink. For the multiple-access relay channel, given the beamforming vectors (51) and the decompression order (94), the transmit power minimization problem subject to the individual rate constraints as well as the individual fronthaul capacity constraints is formulated as

$$\underset{\{p_k^{\text{ul}}\}, \{q_m^{\text{ul}}\}}{\text{minimize}} P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (95)$$

$$\text{subject to } C_{\bar{\rho}(m)}^{\text{ul}, \text{WZ}}(\{p_k^{\text{ul}}\}, q_{\rho^{\text{ul}}(1)}^{\text{ul}}, \dots, q_{\rho^{\text{ul}}(m)}^{\text{ul}}, \{\bar{\rho}(m)\}) \leq C_{\bar{\rho}(m)}, \quad \forall m \in \mathcal{M}, \quad (96)$$

$$(54), (56), (57).$$

Likewise, for the broadcast relay channel as described in Section II-B, given the same transmit beamforming vectors (51) and a reverse compression order (94), the transmit power minimization problem is formulated as

$$\underset{\{p_k^{\text{dl}}\}, \mathbf{Q}}{\text{minimize}} P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}) \quad (97)$$

$$\text{subject to } C_{\bar{\rho}(M+1-m)}^{\text{dl}, \text{MV}}(\{p_k^{\text{dl}}\}, \bar{\mathbf{u}}_k, \mathbf{Q}, \{\bar{\rho}(M+1-m)\}) \leq C_{\bar{\rho}(M+1-m)}, \quad \forall m \in \mathcal{M}, \quad (98)$$

$$\mathbf{Q} \succeq \mathbf{0}, \quad (99)$$

$$(59), (61).$$

If we can show that problem (95) and problem (97) are equivalent, then it implies that under the same fronthaul capacity constraints $\{C_m\}$ and total transmit power constraint P , any achievable rate-tuple in the multiple-access relay channel must also be achievable in the broadcast relay channel by setting the transmit beamforming vectors as the receive beamforming vectors in the multiple-access relay channel and the compression order as the reverse of the decompression order in the multiple-access channel, and vice versa. In addition, by trying all the feasible beamforming vectors, it follows that

$$\bar{\mathcal{R}}_{\text{III}}^{\text{ul}}(\{C_m\}, P, \{\bar{\rho}(m)\}) = \bar{\mathcal{R}}_{\text{III}}^{\text{dl}}(\{C_m\}, P, \{\bar{\rho}(M+1-m)\}). \quad (100)$$

Finally, by trying all the compression/decompression orders, we can show that the achievable rate regions of the multiple-access relay channel under linear decoding as well as Wyner-Ziv compression and the broadcast relay channel under linear encoding as well as multivariate compression are identical, i.e., $\mathcal{R}_{\text{III}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{III}}^{\text{dl}}(\{C_m\}, P)$.

The key to proving Theorem 3 is therefore to show that the power minimization problems (95) and (97) are equivalent. However, different from problems (53) and (58) in Section IV or problems (84) and (86) in Section V, problems (95) and (97) cannot be transformed into an LP due to the complicated expression of the fronthaul rates given in (9) and (33). In the rest of this section, we validate the equivalence between problems (95) and (97) based on Lagrangian duality of SDP. For convenience, we merely consider the case when the decompression order in the multiple-access relay channel and the compression order in the broadcast relay channel are respectively set as

$$\rho^{\text{ul}}(m) = m, \quad \forall m \in \mathcal{M}, \quad (101)$$

$$\rho^{\text{dl}}(m) = M+1-m, \quad \forall m \in \mathcal{M}. \quad (102)$$

For the other decompression order and the corresponding reversed compression order, the equivalence between problems (95) and (97) can be proved in a similar way. The proof involves the following three steps.

A. Convex Reformulation of Problem (97) and Its Dual Problem

First, we transform problem (97) into an equivalent convex problem.

Proposition 4: Power minimization problem (97) in the broadcast relay channel is equivalent to the following problem:

$$\underset{\{p_k^{\text{dl}}\}, \mathbf{Q}}{\text{minimize}} \sum_{k=1}^K p_k^{\text{dl}} \sigma^2 + \text{tr}(\mathbf{Q}) \sigma^2 \quad (103)$$

$$\text{subject to } \frac{p_k^{\text{dl}} |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2R_k - 1} \geq \sum_{j \neq k} p_j^{\text{dl}} |\bar{\mathbf{u}}_j^H \mathbf{h}_k|^2 + \text{tr}(\mathbf{Q} \mathbf{h}_k \mathbf{h}_k^H) + \sigma^2, \quad \forall k \in \mathcal{K}, \quad (104)$$

$$2C_m \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \mathbf{Q}_{(m:M, m:M)} \end{bmatrix}$$

$$- \mathbf{E}_m(\mathbf{Q} + \Psi) \mathbf{E}_m^H \succeq \mathbf{0}, \quad \forall m \in \mathcal{M}, \quad (105)$$

$$(61), (99),$$

$$\text{maximize}_{\{\beta_k\}, \{\Lambda_m\}} \sum_{k=1}^K \beta_k \sigma^2 \quad (106)$$

$$\text{subject to } \sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \geq 0, \quad \forall k \in \mathcal{K}, \quad (107)$$

$$\sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m - \sum_{m=1}^M 2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \Lambda_m^{(m:M, m:M)} \end{bmatrix} \succeq \mathbf{0}, \quad (108)$$

$$\beta_k \geq 0, \quad \forall k \in \mathcal{K}, \quad (109)$$

$$\Lambda_m \succeq \mathbf{0}, \quad \forall m \in \mathcal{M}. \quad (110)$$

$$\text{maximize}_{\{\beta_k\}, \{\Lambda_m, \mathbf{A}_m\}} \sum_{k=1}^K \beta_k \sigma^2 \quad (111)$$

$$\text{subject to } \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m \succeq \mathbf{A}_1, \quad (112)$$

$$\mathbf{A}_m = 2^{C_m} \Lambda_m^{(m:M, m:M)} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-m)} \\ \mathbf{0}_{(M-m) \times 1} & \mathbf{A}_{m+1} \end{bmatrix}, \quad m = 1, \dots, M-1, \quad (113)$$

$$\mathbf{A}_M = 2^{C_M} \Lambda_M^{(M,M)}, \quad (114)$$

(107), (109), (110).

where $\mathbf{E}_m \in \mathbb{C}^{M \times M}$ denotes the matrix where the m -th diagonal element is 1, while the other elements are 0, and $\Psi = \text{diag} \left(\sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,1}|^2, \dots, \sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,M}|^2 \right)$.

Proof: Please refer to Appendix C. ■

It can be seen that problem (103) is an SDP problem and is convex. Further, by our choice of feasible $\{R_k\}$, it satisfies the Slater's condition. As a result, strong duality holds for problem (103). In other words, problem (103) is equivalent to its dual problem. In the following, we derive the dual problem of (103).

Proposition 5: The dual problem of problem (103) is problem (106) at the top of the page, where β_k 's and $\Lambda_m \in \mathbb{C}^{M \times M}$'s are the dual variables associated with constraints (104) and (105), respectively.

Proof: Please refer to Appendix D. ■

It can be observed that similar to the uplink-downlink duality shown in Section IV, the dual problem (106) of the power minimization problem (103) in the broadcast relay channel is closely related to the power minimization problem (95) of the multiple-access relay channel. Specifically, if we interpret the dual variables β_k 's as the uplink transmit powers p_k^{ul} 's and the dual variables $\Lambda_m^{(m,m)}$'s as the uplink quantization noise levels q_m^{ul} 's, then the objective of problem (106) is to maximize the total transmit power, and constraint (107) is to make sure that the rate of each user k in the multiple-access relay channel is no larger than its rate requirement. The remaining challenge is to transform constraint (108) in problem (106) into a set of M fronthaul capacity constraints that are in the same form of constraints (96) in problem (95). Note that in contrast to

the case with independent compression shown in Section IV, where constraint (72) of the dual problem (73) in the broadcast relay channel is directly the reverse of constraint (55) of problem (53) in the multiple-access relay channel, in the case with Wyner-Ziv compression and multivariate compression, the validation of the equivalence between problem (95) and problem (97) is considerably more complicated.

B. Equivalent Transformation of Constraint (108) in the Dual Problem (106)

In the following, we transform constraint (108) in problem (106) into the form of constraint (96) in problem (95). First, we introduce some auxiliary variables to problem (106).

Proposition 6: Problem (106) is equivalent to problem (111) at the top of the page, where $\mathbf{A}_m \in \mathbb{C}^{(M-m+1) \times (M-m+1)}$'s, $m = 1, \dots, M$, are auxiliary variables.

Proof: Please refer to Appendix E. ■

Next, we show some key properties of the optimal solution to problem (111).

Proposition 7: The optimal solution to problem (111) must satisfy

$$\sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m = \mathbf{A}_1, \quad (115)$$

$$\mathbf{A}_m \succ \mathbf{0}, \quad m = 1, \dots, M, \quad (116)$$

$$2^{C_m} \Lambda_m^{(m:M, m:M)} = \frac{\mathbf{A}_m^{(1:M-m+1, 1)} \mathbf{A}_m^{(1, 1:M-m+1)}}{\mathbf{A}_m^{(1,1)}}, \quad \forall m. \quad (117)$$

$$\begin{aligned} & \text{maximize}_{\{\beta_k\}, \{\Lambda_m^{(m,m)}\}} \sum_{k=1}^K \beta_k \sigma^2 \end{aligned} \quad (119)$$

$$\text{subject to } 2^{C_m} \Lambda_m^{(m,m)} \leq \Omega^{(m,m)} - \Omega^{(m,1:m-1)} (\Omega^{(1:m-1,1:m-1)})^{-1} \Omega^{(1:m-1,m)}, \quad \forall m \in \mathcal{M}, \quad (120)$$

$$\begin{aligned} & \Lambda_m^{(m,m)} \geq 0, \quad (121) \\ & (107), (109). \end{aligned}$$

Proof: Please refer to Appendix F. \blacksquare

Proposition 7 indicates that the optimal Λ_m 's to problem (111) are rank-one matrices. This property can be utilized to simplify problem (111) as follows.

Proposition 8: Define

$$\begin{aligned} \Omega &= \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m \\ &= \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \text{diag}(\Lambda_1^{(1,1)}, \dots, \Lambda_M^{(M,M)}) \succ \mathbf{0}. \end{aligned} \quad (118)$$

Then, problem (111) is equivalent to problem (119) at the top of the page.

Proof: Please refer to Appendix G. \blacksquare

Note that the difference between problem (119) and problem (106) are two-fold. First, the optimization variables reduce from Λ_m 's to their m -th diagonal elements, i.e., $\Lambda_m^{(m,m)}$'s. Second, constraint (108) in the matrix form reduces to M constraints given in (120) in the scalar form.

If we interpret the dual variables β_k 's as the transmit powers p_k^{ul} 's and the dual variables $\Lambda_m^{(m,m)}$'s as the quantization noise levels q_m^{ul} 's in the multiple-access relay channel, then problem (119) is equivalent to the maximization of the transmit power subject to the constraints that the rate of each user k is no larger than R_k , $\forall k$, and the fronthaul rate of each relay m is no smaller than C_m , $\forall m$. As a result, problem (119) is a reverse problem to the power minimization problem (95) in the multiple-access relay channel. In the following, we show that this problem is indeed equivalent to the power minimization problem (95).

C. Equivalence Between Power Maximization Problem (119) and Power Minimization Problem (95)

First, we prove one important property of the optimal solution to problem (119).

Proposition 9: The optimal solution to problem (119) must have its constraints (107) and (120) satisfied with equality, i.e.,

$$\begin{aligned} \sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \\ = 0, \quad \forall k \in \mathcal{K}, \end{aligned} \quad (122)$$

$$\begin{aligned} 2^{C_m} \Lambda_m^{(m,m)} &= \\ \Omega^{(m,m)} - \Omega^{(m,1:m-1)} (\Omega^{(1:m-1,1:m-1)})^{-1} \Omega^{(1:m-1,m)}, \\ &\quad \forall m \in \mathcal{M}. \end{aligned} \quad (123)$$

Proof: Please refer to Appendix H. \blacksquare

According to Proposition 9, problem (119) is equivalent to the following problem

$$\begin{aligned} & \text{maximize}_{\{\beta_k\}, \{\Lambda_m^{(m,m)}\}} \sum_{k=1}^K \beta_k \sigma^2 \quad (124) \\ & \text{subject to } (109), (121), (122), (123). \end{aligned}$$

Next, we show one important property of the optimal solution to problem (124).

Proposition 10: If there exists one set of solutions $\{\beta_k, \Lambda_m^{(m,m)}\}$ that satisfies the constraints (109), (121), (122), and (123) in problem (124), then this solution is unique.

Proof: Please refer to Appendix I. \blacksquare

Note that the above proposition is very similar to Proposition 2 for the case with independent compression shown in Section IV. However, the proof of Proposition 10 is much more complicated. This is because for the case with independent compression, equations in (74) and (75) are linear in terms of β_k 's and λ_m 's, while for the case with Wyner-Ziv compression and multivariate compression, equations in (123) are not linear in terms of β_k 's and $\Lambda_m^{(m,m)}$'s.

Since there is a unique solution that satisfies all the constraints in problem (124), it follows that maximization problem (124) is equivalent to the following minimization problem

$$\begin{aligned} & \text{minimize}_{\{\beta_k\}, \{\Lambda_m^{(m,m)}\}} \sum_{k=1}^K \beta_k \sigma^2 \quad (125) \\ & \text{subject to } (109), (121), (122), (123). \end{aligned}$$

Last, we relate problem (125) to the power minimization problem (95) in the multiple-access relay channel by the following proposition.

Proposition 11: Problem (125) is equivalent to problem (126) at the top of the next page.

Proof: Similar to Proposition 9, it can be shown that the optimal solution to problem (126) must have its constraints, shown as (127) and (128) on top of the next page, satisfied with equality. As a result, problem (125) is equivalent to problem (126). Proposition 11 is thus proved. \blacksquare

In problem (126), the dual variables β_k 's can be viewed as the transmit powers p_k^{ul} 's, and the dual variables $\Lambda_m^{(m,m)}$'s can be viewed as the quantization noise levels q_m^{ul} 's in the multiple-access relay channel. Moreover, according to (10) and (118), Ω can be viewed as the covariance matrix of the received signal in the multiple-access relay channel, i.e., Γ as defined by (10). As a result, by combining Propositions 4 – 11, we can conclude that the power minimization problem (97) for the

$$\underset{\{\beta_k\}, \{\mathbf{\Lambda}_m^{(m,m)}\}}{\text{minimize}} \quad \sum_{k=1}^K \beta_k \sigma^2 \quad (126)$$

$$\text{subject to} \quad \sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \mathbf{\Lambda}_m^{(m,m)} |\bar{u}_{k,m}|^2 - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \leq 0, \quad \forall k \in \mathcal{K}, \quad (127)$$

$$2^{C_m} \mathbf{\Lambda}_m^{(m,m)} \geq \mathbf{\Omega}^{(m,m)} - \mathbf{\Omega}^{(m,1:m-1)} (\mathbf{\Omega}^{(1:m-1,1:m-1)})^{-1} \mathbf{\Omega}^{(1:m-1,m)}, \quad \forall m \in \mathcal{M}. \quad (128)$$

(109), (121).

broadcast relay channel is equivalent to problem (126), and is thus equivalent to power minimization problem (95) for the multiple-access relay channel. Theorem 3 is thus proved.

To summarize the difference in the methodology for proving Theorem 1 and Theorem 3, we note that although the proof of Theorem 3 for the case of Wyner-Ziv compression in the multiple-access relay channel and multivariate compression in the broadcast relay channel follows the same line of reasoning as the proof of Theorem 1 for the case with independent compression (i.e., we first use the Lagrangian duality method to find the dual problem of the power minimization problem in the broadcast relay channel, then show that this dual problem is equivalent to the power minimization problem in the multiple-access relay channel), the validation of the equivalence between the dual problem in the broadcast relay channel and the power minimization problem in the multiple-access channel is much more involved for the case of Wyner-Ziv and multivariate compression. This is because (i) the downlink power minimization problem becomes an SDP, rather than an LP for the case of independent compression, and the duality of SDPs are more complicated than that of LPs; (ii) the step in Section VI-B is needed, since constraint (108) in the dual problem (106) is not a direct reverse of constraint (96) in problem (95); (iii) Proposition 10 involves nonlinear equations, and it is more difficult to show that its solution is unique.

VII. PROOF OF THEOREM 4

In this section, we prove the duality between the multiple-access relay channel with successive interference cancellation at the CP as well as Wyner-Ziv compression across the relays and the broadcast relay channel with dirty-paper coding at the CP as well as multivariate compression across the relays.

Similar to Sections IV, V, and VI, we fix the same beamforming vectors $\bar{\mathbf{u}}_k$'s in the multiple-access relay channel and the broadcast relay channel as shown in (51), where $\bar{\mathbf{u}}_k$'s satisfy (52). Next, for successive interference cancellation at the CP in the multiple-access relay channel and dirty-paper coding at the CP in the broadcast relay channel, we assume that the decoding order is the reverse of the encoding order, i.e., (83). For example, if the decoding order in the multiple-access relay channel is $1, \dots, K$, then the encoding order in the broadcast relay channel is $K, \dots, 1$. Moreover, for Wyner-Ziv compression across the relays in the multiple-access relay channel and multivariate compression across the relays in the broadcast relay channel, we assume the decompression order is the reverse of the compression order, i.e., (94). For example,

if the decompression order in the multiple-access relay channel is $1, \dots, M$, then the compression order in the broadcast relay channel is $M, \dots, 1$.

Let $\{R_k > 0, k \in \mathcal{K}\}$ and $\{C_m > 0, m \in \mathcal{M}\}$ denote sets of strictly feasible user rate requirements and fronthaul rate requirements under the beamforming vectors (51), decoding order (83), and decompression order (94) for both the uplink and the downlink. For the multiple-access relay channel, the transmit power minimization problem subject to the individual rate constraints as well as the individual fronthaul capacity constraints is formulated as

$$\underset{\{p_k^{\text{ul}}\}, \{q_m^{\text{ul}}\}}{\text{minimize}} \quad P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (129)$$

subject to (56), (57), (85), (96).

Likewise, given the transmit beamforming vectors (51), a reversed encoding order (83), and a reversed compression order (94), for the broadcast relay channel, the transmit power minimization problem is formulated as

$$\underset{\{p_k^{\text{dl}}\}, \mathbf{Q}}{\text{minimize}} \quad P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}) \quad (130)$$

subject to (61), (62), (87), (98).

Similar to Section VI, we can apply the Lagrangian duality method to show that problem (130) is equivalent to problem (129). Since the method adopted is almost the same as that in Section VI, we omit the proof here. As a result, under the same fronthaul capacity constraints C_m 's and total transmit power constraint P , any achievable rate-tuple for the multiple-access relay channel is also achievable for the broadcast relay channel by setting the transmit beamforming vectors as the receive beamforming vectors in the multiple-access relay channel and by setting the encoding and compression order to be the reverse of the decoding and decompression order in the multiple-access relay channel, and vice versa. In other words, by trying all the feasible beamforming vectors, it follows that

$$\begin{aligned} & \bar{\mathcal{R}}_{\text{IV}}^{\text{ul}}(\{C_m\}, P, \{\bar{\rho}(m)\}, \{\bar{\tau}(k)\}) \\ &= \bar{\mathcal{R}}_{\text{IV}}^{\text{dl}}(\{C_m\}, P, \{\bar{\rho}(M+1-m)\}, \{\bar{\tau}(K+1-k)\}). \end{aligned} \quad (131)$$

Then, by trying all the encoding/decoding orders and compression/decompression orders, we can show that the achievable rate regions of the multiple-access relay channel under successive interference cancellation as well as Wyner-Ziv compression and the broadcast relay channel under dirty-paper

TABLE I
PRIMAL AND DUAL RELATIONSHIPS BETWEEN THE SUM-POWER MINIMIZATION PROBLEMS FOR THE MULTIPLE-ACCESS RELAY CHANNEL AND THE BROADCAST RELAY CHANNEL

	Broadcast relay channel	Multiple-access relay channel
	Fixing beamformers $\{\mathbf{w}_k = \mathbf{v}_k\}$	
Case I	Primal problem: Power minimization (58) • Rate constraint (59): Optimal dual variables $\{\beta_k\}$ • Fronthaul constraint (60): Optimal dual variables $\{\lambda_m\}$	Dual problem: Power minimization (53) • Optimal transmit powers: $\{p_k^{\text{ul}} = \beta_k\}$ • Optimal quantization noises: $\{q_m^{\text{ul}} = \lambda_m\}$
Case II	Primal problem: Power minimization (86) • Rate constraint (87): Optimal dual variables $\{\beta_k\}$ • Fronthaul constraint (60): Optimal dual variables $\{\lambda_m\}$	Dual problem: Power minimization (84) • Optimal transmit powers: $\{p_k^{\text{ul}} = \beta_k\}$ • Optimal quantization noises: $\{q_m^{\text{ul}} = \lambda_m\}$
Case III	Primal problem: Power minimization (97) • Rate constraint (59): Optimal dual variables $\{\beta_k\}$ • Fronthaul constraint (105): Optimal dual variables $\{\Lambda_m\}$	Dual problem: Power minimization (95) • Optimal transmit powers: $\{p_k^{\text{ul}} = \beta_k\}$ • Optimal quantization noises: $\{q_m^{\text{ul}} = \Lambda_m^{(m,m)}\}$
Case IV	Primal problem: Power minimization (130) • Rate constraint (87): Optimal dual variables $\{\beta_k\}$ • Fronthaul constraint (105): Optimal dual variables $\{\Lambda_m\}$	Dual problem: Power minimization (129) • Optimal transmit powers: $\{p_k^{\text{ul}} = \beta_k\}$ • Optimal quantization noises: $\{q_m^{\text{ul}} = \Lambda_m^{(m,m)}\}$

coding as well as multivariate compression are identical, i.e., $\mathcal{R}_{\text{IV}}^{\text{ul}}(\{C_m\}, P) = \mathcal{R}_{\text{IV}}^{\text{dl}}(\{C_m\}, P)$.

VIII. DUALITY RELATIONSHIPS

The main result of this paper is that the Lagrangian duality technique can be used to establish the duality between the sum-power minimization problems for the multiple-access relay channel and the broadcast relay channel. The key observation is that under fixed beamformers, the sum-power minimization problems in both the uplink and the downlink can be transformed into convex optimization problems. In particular, with independent compression (i.e., Cases I and II), both the uplink and downlink problems are LPs, while with Wyner-Ziv or multivariate compression (i.e., Cases III and IV), both the uplink and downlink problems can be transformed into SDPs. Note that for the uplink problem under the Wyner-Ziv compression, the transformation into SDP also involves reversing the minimization into the maximization and reversing the direction of inequalities involving the rate and fronthaul constraints. We further show that the resulting convex problems in the uplink and the downlink after the transformations are the Lagrangian duals of each other for Cases I–IV. Moreover, the dual variables have interesting physical interpretations. Specifically, the Lagrangian dual variables corresponding to the downlink achievable rate constraints are the optimal uplink transmit powers; the dual variables corresponding to the downlink fronthaul rate constraints are the optimal uplink quantization noise levels. These interpretations are summarized in Table I.

In the prior literature, the traditional uplink-downlink duality relationship is established by showing that any achievable rate-tuple in the uplink is also achievable in the downlink, and vice versa. However, it is difficult to apply this approach to verify the duality results of this paper, at least for the case with the Wyner-Ziv and multivariate compression strategies. This is because the condition to ensure a feasible solution is easy to characterize only under linear constraints (i.e., with independent compression). But with Wyner-Ziv and multivariate compressions, the fronthaul rates are nonlinear functions of the transmit powers and the quantization noises. For this reason, this paper takes an alternative approach of

fixing the beamforming vectors then transforming the sum-power minimization problems subject to the individual rate and fronthaul constraints into suitable convex forms. As a result, we are able to take a unified approach to establish the uplink-downlink duality using the Lagrangian duality theory. This approach in fact works for both independent compression and multivariate compression.

IX. APPLICATION OF DUALITY

While the main technical proofs in this paper are for the case of fixed beamformers, the duality relationship also extends to the scenario where the beamformers need to be jointly optimized with the transmit powers and quantization noises. In this section, we consider algorithms for such joint optimization problems and show that the duality relationship gives an efficient way of solving the downlink joint optimization problem via its uplink counterpart.

The joint optimization of the beamforming vectors, transmit powers, and the quantization noises for the broadcast relay channel is more difficult than the corresponding optimization for the multiple-access relay channel. This is because in the multiple-access relay channel, the receive beamforming vectors $\{\mathbf{w}_k\}$ affect the user rates only, but do not affect the fronthaul rates; the optimal receive beamformers are simply the minimum mean-squared-error (MMSE) receiver. This is in contrast to the broadcast relay channel, where the transmit beamforming vectors $\{\mathbf{v}_k\}$ affect both the user rates and the fronthaul rates, which makes the optimization highly nontrivial. Furthermore, the optimization of the quantization noise solution is also conceptually easier in the multiple-access relay channel since the quantization noise levels are scalars, while in the broadcast relay channel, the optimization of the quantization noises is over their covariance matrix. Due to the above reasons, it would be appealing if we can solve the broadcast relay channel problem via its dual multiple-access counterpart. In the following two subsections, we show how this can be done under independent compression and Wyner-Ziv/multivariate compression, respectively. The key observation is that in both cases, the sum-power minimization problem in the multiple-access relay channel can be solved globally via a fixed-point iteration method.

$$I_k(\mathbf{p}^{\text{ul}}) = \frac{2^{R_k} - 1}{\mathbf{h}_k^H \left(\sum_{j \neq k} p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}_k}. \quad (143)$$

A. Independent Compression

First, consider the sum-power minimization problems in the multiple-access and broadcast relay channels for Cases I and II with independent compression/decompression. For simplicity, we focus on Case I, i.e., linear encoding/decoding, and show how to solve the problem in the broadcast relay channel via solving the dual problem in the multiple-access relay channel. Similar approach can be applied to Case II under nonlinear encoding/decoding. Specifically, in Case I, the sum-power minimization problems in the multiple-access channel and the broadcast relay channel are formulated as

$$\underset{\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}}{\text{minimize}} \quad P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (132)$$

$$\text{subject to } R_k^{\text{ul}, \text{TIN}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) \geq R_k, \quad \forall k \in \mathcal{K}, \quad (133)$$

$$C_m^{\text{ul}, \text{LN}}(\{p_k^{\text{ul}}\}, q_m^{\text{ul}}) \leq C_m, \quad \forall m \in \mathcal{M}, \quad (134)$$

(56), (57),

and

$$\underset{\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}}{\text{minimize}} \quad P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}_{\text{diag}}) \quad (135)$$

$$\text{subject to } R_k^{\text{dl}, \text{LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) \geq R_k, \quad \forall k \in \mathcal{K}, \quad (136)$$

$$C_m^{\text{dl}, \text{LN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}_{\text{diag}}) \leq C_m, \quad \forall m \in \mathcal{M}, \quad (137)$$

(61), (62).

Note that different from problems (53) and (58), here the beamforming vectors need to be jointly optimized with the transmit powers and quantization noises. In the following, we propose a fixed-point iteration method that can globally solve problem (132) with low complexity. The main idea is to transform problem (132) that jointly optimizes the transmit powers, receive beamforming vectors, and quantization noise levels into a power control problem. Specifically, according to Proposition 3, at the optimal solution, constraint (134) in problem (132) should be satisfied with equality. In this case, the fronthaul capacity constraint (134) gives the following relationship between the optimal quantization noise levels and the optimal transmit powers:

$$q_m^{\text{ul}}(\mathbf{p}^{\text{ul}}) = \frac{\sum_{k=1}^K p_k^{\text{ul}} |h_{m,k}|^2 + \sigma^2}{2C_m - 1}, \quad \forall m \in \mathcal{M}, \quad (138)$$

where $\mathbf{p}^{\text{ul}} = [p_1^{\text{ul}}, \dots, p_K^{\text{ul}}]^T$. Moreover, it is observed from problem (132) that the receive beamforming vectors affect the user rates only, but do not affect the fronthaul rates. It

is well-known that the optimal receive beamforming vectors to maximize the user rates are the MMSE receivers, i.e.,

$$\mathbf{w}_k = \frac{\tilde{\mathbf{w}}_k}{\|\tilde{\mathbf{w}}_k\|}, \quad \forall k \in \mathcal{K}, \quad (139)$$

where

$$\tilde{\mathbf{w}}_k = \left(\sum_{j \neq k} p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}_k, \quad (140)$$

with $q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$ as given in (138). By plugging the above MMSE beamformers into constraint (133), problem (132) is now equivalent to the following power control problem:

$$\underset{\mathbf{p}^{\text{ul}}}{\text{minimize}} \quad P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (141)$$

$$\text{subject to } p_k^{\text{ul}} \geq I_k(\mathbf{p}^{\text{ul}}), \quad \forall k \in \mathcal{K}, \quad (142)$$

where $\forall k \in \mathcal{K}$, and $I_k(\mathbf{p}^{\text{ul}})$ is expressed in (143) at the top of the page.

Lemma 1: Given $\mathbf{p}^{\text{ul}} \geq \mathbf{0}$, the functions $I_k(\mathbf{p}^{\text{ul}})$'s, $k = 1, \dots, K$, as defined by (143) satisfy the following three properties:

1. $I_k(\mathbf{p}^{\text{ul}}) > 0, \forall k$;
2. Given any $\alpha > 1$, it follows that $I_k(\alpha \mathbf{p}^{\text{ul}}) < \alpha I_k(\mathbf{p}^{\text{ul}}), \forall k$;
3. If $\bar{\mathbf{p}}^{\text{ul}} \geq \mathbf{p}^{\text{ul}}$, then $I_k(\bar{\mathbf{p}}^{\text{ul}}) \geq I_k(\mathbf{p}^{\text{ul}}), \forall k$.

Proof: Please refer to Appendix J. ■

Lemma 1 shows that $\mathbf{I}(\mathbf{p}^{\text{ul}}) = [I_1(\mathbf{p}^{\text{ul}}), \dots, I_K(\mathbf{p}^{\text{ul}})]^T$ is a standard interference function [49]. According to [49, Theorem 2], as long as the original problem is feasible, given any initial power solution $\mathbf{p}^{\text{ul},(0)} = [p_1^{\text{ul},(0)}, \dots, p_K^{\text{ul},(0)}]^T$ that satisfies $\mathbf{p}^{\text{ul},(0)} \geq \mathbf{0}$, the following fixed-point iteration must converge to the globally optimal power control solution to problem (141):

$$p_k^{\text{ul},(t+1)} = I_k(\mathbf{p}^{\text{ul},(t)}), \quad \forall k \in \mathcal{K}, \quad (144)$$

where $I_k(\mathbf{p}^{\text{ul}})$'s are defined in (143), and $\mathbf{p}^{\text{ul},(t)}$ is the power obtained after the t -th iteration of the fixed-point update (144). After the optimal power solution denoted by $\mathbf{p}^{\text{ul},*}$ is obtained via the fixed-point iteration, the optimal quantization noise levels denoted by $\{q_m^{\text{ul},*}\}$ can be obtained via (138), and the optimal receive beamforming vectors denoted by $\{\mathbf{w}_k^*\}$ can be obtained via (139). Note that the above algorithm to solve problem (132) is very simple, because all the updates have closed-form expressions.

After problem (132) is solved globally, we now show how to obtain the optimal solution to problem (135) using the uplink-downlink duality. It is shown in Section IV that given the same

$$q_m^{\text{ul}}(\mathbf{p}^{\text{ul}}) = \frac{\sigma^2 + \sum_{k=1}^K p_k^{\text{ul}} h_{k,m} h_{k,m}^H - \mathbf{\Gamma}^{(m,1:m-1)}(\mathbf{p}^{\text{ul}}) (\mathbf{\Gamma}^{(1:m-1,1:m-1)}(\mathbf{p}^{\text{ul}}))^{-1} \mathbf{\Gamma}^{(1:m-1,m)}(\mathbf{p}^{\text{ul}})}{2C_m - 1}, \quad \forall m \in \mathcal{M}. \quad (155)$$

beamforming vectors, the sum-power minimization problems in the multiple-access channel and the broadcast relay channel are equivalent to each other. As a result, the optimal beamforming vectors for problem (132), i.e., $\{\mathbf{w}_k^*\}$, are also the optimal beamforming vectors for problem (135). After setting $\mathbf{v}_k^* = \mathbf{w}_k^*, \forall k$, the problem reduces to an optimization problem over the transmit powers and quantization noise levels. As problem (135) is convex in p_k^{dl} and \mathbf{Q}_{diag} given the fixed beamforming vectors as shown in Section IV, it can be solved efficiently.

We contrast the above approach with the direct optimization of the downlink. It can be shown that if the unit-power beamformers \mathbf{v}_k 's and transmit powers p_k^{dl} 's are combined together as $\tilde{\mathbf{v}}_k = p_k^{\text{dl}} \mathbf{v}_k = [\tilde{v}_{k,1}, \dots, \tilde{v}_{k,M}]^T, \forall k$, then problem (135) can be transformed into the following optimization problem:

$$\text{minimize}_{\{\tilde{\mathbf{v}}_k\}, \{q_m^{\text{dl}}\}} \sum_{k=1}^K \|\tilde{\mathbf{v}}_k\|^2 \sigma^2 + \sum_{m=1}^M q_m^{\text{dl}} \sigma^2 \quad (145)$$

$$\text{subject to } \frac{|\tilde{\mathbf{v}}_k^H \mathbf{h}_k|^2}{2R_k - 1} - \sum_{j \neq k} |\tilde{\mathbf{v}}_j^H \mathbf{h}_k|^2 - \sum_{m=1}^M q_m^{\text{dl}} |h_{k,m}|^2 - \sigma^2 \geq 0, \quad \forall k \in \mathcal{K}, \quad (146)$$

$$\sum_{k=1}^K |\tilde{v}_{k,m}|^2 + q_m^{\text{dl}} - 2^{C_m} q_m^{\text{dl}} \leq 0, \quad \forall m \in \mathcal{M}, \quad (147)$$

$$(61), (62).$$

Note that (146) can be transformed into a second-order cone constraint [8]. As a result, problem (145) for the broadcast relay channel is convex. However, the joint optimization of the beamforming vectors and quantization noise levels in problem (145) is over a higher dimension, so it can potentially be more computationally complex than the duality-based approach where the beamforming vectors are first obtained by solving problem (132) using fixed-point iterations, then the powers and quantization noise levels are optimized given the beamforming vectors.

B. Wyner-Ziv and Multivariate Compression

The above approach can also be used for the scenarios with Wyner-Ziv and multivariate compressions, i.e., Case III under linear encoding/decoding and Case IV under nonlinear encoding/decoding. For simplicity, in the following, we focus on Case III; similar analysis also applies to Case IV. Suppose that the decompression order for Wyner-Ziv compression and the compression order for multivariate compression are given by (94). Then, the sum-power minimization problems under

Case III in the multiple-access relay channel and the broadcast relay channel are respectively given by

$$\begin{aligned} & \text{minimize}_{\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}} P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (148) \\ & \text{subject to } R_k^{\text{ul}, \text{TIN}}(\{p_k^{\text{ul}}, \mathbf{w}_k\}, \{q_m^{\text{ul}}\}) \geq R_k, \quad \forall k \in \mathcal{K}, \quad (149) \end{aligned}$$

$$\begin{aligned} & C_{\bar{\rho}(m)}^{\text{ul}, \text{WZ}}(\{p_k^{\text{ul}}\}, q_{\bar{\rho}(1)}^{\text{ul}}, \dots, q_{\bar{\rho}(m)}^{\text{ul}}, \{\bar{\rho}(m)\}) \\ & \leq C_{\bar{\rho}(m)}, \quad \forall m \in \mathcal{M}, \quad (150) \end{aligned}$$

$$(56), (57),$$

and

$$\text{minimize}_{\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}} P^{\text{dl}}(\{p_k^{\text{dl}}\}, \mathbf{Q}) \quad (151)$$

$$\text{subject to } R_k^{\text{dl}, \text{LIN}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}) \geq R_k, \quad \forall k \in \mathcal{K}, \quad (152)$$

$$\begin{aligned} & C_{\bar{\rho}(M+1-m)}^{\text{dl}, \text{MV}}(\{p_k^{\text{dl}}, \mathbf{v}_k\}, \mathbf{Q}, \{\bar{\rho}(M+1-m)\}) \\ & \leq C_{\bar{\rho}(M+1-m)}, \quad \forall m \in \mathcal{M}, \quad (153) \end{aligned}$$

$$(99), (61).$$

Next, we show how to solve problem (151) via problem (148). Similar to problem (132), in the following, we transform problem (148) into a power control problem, then propose a fixed-point iteration method to solve this power control problem globally. First, according to Proposition 11, with the optimal solution to problem (148), constraint (150) in problem (148) is satisfied with equality. In this case, the quantization noise level of relay 1 can be expressed as a function of the transmit powers as follows:

$$q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}) = \frac{\sigma^2 + \sum_{k=1}^K p_k^{\text{ul}} h_{k,1} h_{k,1}^H}{2C_1 - 1}. \quad (154)$$

Moreover, if $q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_{m-1}^{\text{ul}}(\mathbf{p}^{\text{ul}})$ are already expressed as functions of \mathbf{p}^{ul} , then $\mathbf{\Gamma}^{(1:m-1,1:m-1)}(\mathbf{p}^{\text{ul}})$, $\mathbf{\Gamma}^{(m,1:m-1)}$, and $\mathbf{\Gamma}^{(1:m-1,m)}$ can also be expressed as functions of \mathbf{p}^{ul} according to (10), given the specific decompression order (94). For convenience, we use $\mathbf{\Gamma}^{(1:m-1,1:m-1)}(\mathbf{p}^{\text{ul}})$, $\mathbf{\Gamma}^{(m,1:m-1)}(\mathbf{p}^{\text{ul}})$, and $\mathbf{\Gamma}^{(1:m-1,m)}(\mathbf{p}^{\text{ul}})$ to indicate that they are functions of \mathbf{p}^{ul} . In this case, $q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$ can be uniquely expressed as a function of \mathbf{p}^{ul} as given in (155) on top of the page. In other words, given \mathbf{p}^{ul} , we can first characterize $q_1^{\text{ul}}(\mathbf{p}^{\text{ul}})$ according to (154), then characterize $q_2^{\text{ul}}(\mathbf{p}^{\text{ul}})$ given $q_1^{\text{ul}}(\mathbf{p}^{\text{ul}})$ according to (155), then characterize $q_3^{\text{ul}}(\mathbf{p}^{\text{ul}})$ given $q_1^{\text{ul}}(\mathbf{p}^{\text{ul}})$ and $q_2^{\text{ul}}(\mathbf{p}^{\text{ul}})$ according to (155), and so on.

Next, similar to problem (132), given the transmit power solution \mathbf{p}^{ul} and the quantization noise levels $\{q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})\}$ as in (154) and (155), the optimal MMSE beamforming vectors

$$I_k(\mathbf{p}^{\text{ul}}) = \frac{2^{R_k} - 1}{\mathbf{h}_k^H \left(\sum_{j \neq k} p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}_k}, \quad \forall k \in \mathcal{K}. \quad (158)$$

are given in (139). By plugging them into problem (148), we have the following power control problem:

$$\underset{\mathbf{p}^{\text{ul}}}{\text{minimize}} \quad P^{\text{ul}}(\{p_k^{\text{ul}}\}) \quad (156)$$

$$\text{subject to} \quad p_k^{\text{ul}} \geq I_k(\mathbf{p}^{\text{ul}}), \quad \forall k \in \mathcal{K}, \quad (157)$$

where $\{I_k(\mathbf{p}^{\text{ul}})\}$ are given in (158) at the top of the page and $\{q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})\}$ are given in (154) and (155).

Lemma 2: Given $\mathbf{p}^{\text{ul}} \geq \mathbf{0}$, the functions $I_k(\mathbf{p}^{\text{ul}})$'s, $k = 1, \dots, K$, defined by (158) satisfy the following three properties:

1. $I_k(\mathbf{p}^{\text{ul}}) > 0, \forall k$;
2. Given any $\alpha > 1$, it follows that $I_k(\alpha \mathbf{p}^{\text{ul}}) < \alpha I_k(\mathbf{p}^{\text{ul}}), \forall k$;
3. If $\bar{\mathbf{p}}^{\text{ul}} \geq \mathbf{p}^{\text{ul}}$, then $I_k(\bar{\mathbf{p}}^{\text{ul}}) \geq I_k(\mathbf{p}^{\text{ul}}), \forall k$.

Proof: Please refer to Appendix K. \blacksquare

Similar to Lemma 1, Lemma 2 shows that $\mathbf{I}(\mathbf{p}^{\text{ul}}) = [I_1(\mathbf{p}^{\text{ul}}), \dots, I_K(\mathbf{p}^{\text{ul}})]^T$ is a standard interference function [49]. According to [49, Theorem 2], as long as the original problem is feasible, given any initial power solution $\mathbf{p}^{\text{ul},(0)} = [p_1^{\text{ul},(0)}, \dots, p_K^{\text{ul},(0)}]^T$ that satisfies $\mathbf{p}^{\text{ul},(0)} \geq \mathbf{0}$, the following fixed-point iteration must converge to the globally optimal power control solution to problem (156):

$$p_k^{\text{ul},(t+1)} = I_k(\mathbf{p}^{\text{ul},(t)}), \quad \forall k \in \mathcal{K}, \quad (159)$$

where $I_k(\mathbf{p}^{\text{ul}})$'s are defined in (158), and $\mathbf{p}^{\text{ul},(t)}$ denotes the power solution obtained at the t -th iteration of the fixed-point update (159). To implement the above fixed-point iteration, given $\mathbf{p}^{\text{ul},(t)}$, we can first get $q_1^{\text{ul}}(\mathbf{p}^{\text{ul},(t)})$ according to (154), then $q_2^{\text{ul}}(\mathbf{p}^{\text{ul},(t)})$ given $q_1^{\text{ul}}(\mathbf{p}^{\text{ul},(t)})$ according to (155), and so on. Thus, given $\mathbf{p}^{\text{ul},(t)}$, we get $\{q_m^{\text{ul}}(\mathbf{p}^{\text{ul},(t)})\}$, then $\{I_k(\mathbf{p}^{\text{ul},(t)})\}$ in (159) can be computed by using (158).

After the optimal power $\mathbf{p}^{\text{ul},*}$ is obtained by the fixed-point method (159), the optimal quantization noise levels $\{q_m^{\text{ul},*}\}$ can then be obtained via (154) and (155), and the optimal receive beamforming vectors $\{\mathbf{w}_k^*\}$ can be obtained via (139). Note that the above algorithm to solve problem (148) is simple, because there are closed-form expressions for the update of all the variables.

Finally, we solve problem (151) via duality. According to the duality result in Section VI, we can set the transmit beamforming vectors in problem (151) as $\mathbf{v}_k^* = \mathbf{w}_k^*, \forall k \in \mathcal{K}$. Then, we can obtain the optimal transmit power and quantization covariance matrix given this beamforming solution by solving the convex problem (103). Note that it is not yet known whether problem (151) has a convex reformulation. Nevertheless, the above algorithm gives a globally optimal solution for (151). As a final remark, it is worth emphasizing that a condition for the convergence of the fixed-point iteration algorithm is that the original problem is feasible to start with.

However, determining whether a set of user target rates and fronthaul capacities is feasible is by itself not necessarily always easy to do computationally, unless the problem can be reformulated as a convex optimization.

C. Numerical Example

As a numerical example demonstrating the algorithm of using uplink-downlink duality to solve the broadcast relay channel problem, we consider a network with $M = 3$ relays and $K = 3$ users, where the wireless channels between these relays and users are generated based on the independent and identically distributed Rayleigh fading model with zero mean and unit variance, and the fronthaul capacities between all the relays and the CP are set to be 3 bps. Moreover, the noise powers at the relays in the uplink and at the users in the downlink are set to be $\sigma^2 = 1$. The rate targets for all the users are assumed to be identical. Under the above setup, Fig. 3 shows the optimal values of the uplink (UL) problem (132) and the downlink (DL) problem (135) under independent compression (IN), and respectively the UL problem (148) and the DL problem (151) under the Wyner-Ziv (WZ) and multivariate (MV) compression, as functions of the user rate target, for both the cases of treating interference as noise (TIN) and linear encoding (LIN), respectively. In addition, the corresponding curves for the cases of successive interference cancellation (SIC) in the uplink or dirty-paper coding (DPC) in the downlink are also plotted. The numerical values are obtained under the optimal beamforming vectors computed based on the multiple-access relay channel; the same beamformers are also optimal for the broadcast relay channel. Under these fixed beamformers, the numerical results show that for both the cases of independent compression and Wyner-Ziv/multivariate compression, at the same feasible rate targets, the minimum sum power in the broadcast relay channel is the same as that in the multiple-access relay channel. Moreover, as a consequence of this duality relationship, the ranges of the feasible rate targets are also the same for the broadcast relay channel and the multiple-access relay channel, under various encoding/decoding and compression/decompression strategies. For example, under the case of linear encoding/decoding and independent compression/decompression, if the common rate target is beyond about 2.4 bps, the sum power minimization problem becomes infeasible in both the uplink and the downlink. It is worth noting that the minimum sum power under the Wyner-Ziv and multivariate compression is smaller than that under the independent compression, especially when the user rate target is high. This is because the Wyner-Ziv compression can utilize the fronthaul more efficiently by using the decompressed signals as side information, while the multivariate compression can reduce the interference caused

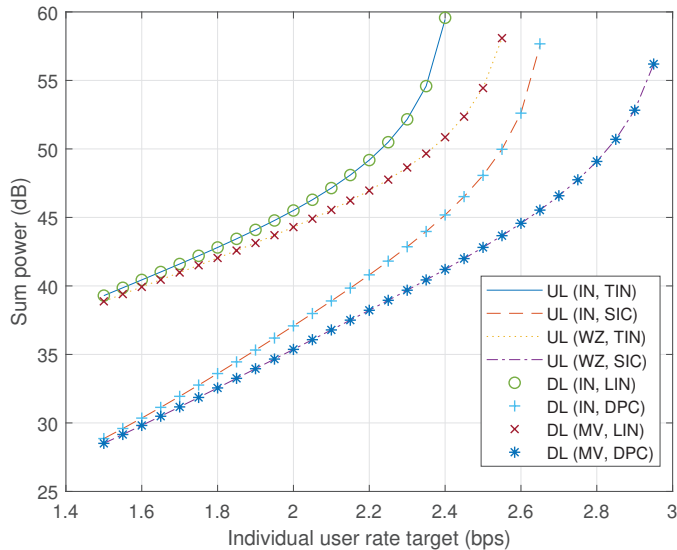


Fig. 3. Illustration of the numerical algorithm for using duality to solve the downlink sum-power minimization problem via the dual uplink problem.

by compression as seen by the users. Finally, successive decoding and dirty-paper coding have significant benefit as compared to treating interference as noise and linear precoding, respectively.

X. CONCLUDING REMARKS

This paper reveals an interesting duality relationship between the Gaussian multiple-access relay channel and the Gaussian broadcast relay channel. Specifically, we first show that if independent compression is applied in both the uplink and the downlink, then the achievable rate regions of the multiple-access relay channel and the broadcast relay channel are identical under the same total transmit power constraint and individual fronthaul capacity constraints. Furthermore, this duality continues to hold when the Wyner-Ziv compression strategy is applied in the multiple-access relay channel and the multivariate compression strategy is applied in the broadcast relay channel. This duality relationship has an intimate connection to the Lagrangian duality theory in convex optimization. Under fixed beamformers, the power minimization problems for the multiple-access and broadcast relay channels are the Lagrangian duals of each other. The optimal dual variables corresponding to the rate constraints in the broadcast relay channel problem are the optimal transmit powers in the dual multiple-access relay channel problem. The optimal dual variables corresponding to the fronthaul constraints in the broadcast relay channel problem are the optimal quantization noise levels in the dual multiple-access relay channel problem. Furthermore, this paper shows that for jointly optimizing over the transmit powers, the receive beamforming vectors, and the quantization noise levels, the uplink sum-power minimization problem can be globally solved using a low-complexity fixed-point iteration method. Thus, the duality also gives a computationally simple way of finding the optimal beamformers in the downlink problem via its dual uplink.

We remark that the duality relationship established in this paper is specific to the compression based relaying strategies. The compression strategies considered in this paper are special cases of the more general noisy network coding [44] and distributed decode-forward [45] schemes for general multi-hop relay networks. An *operational* duality between noisy network coding and distributed decode-forward has already been observed in [45], thus the main contribution of this paper is in establishing a *computational* duality for the specific compression schemes in the setting of the specific two-hop relay network. Is there a *computational* duality for the more general network? Is there a *capacity region* duality for either the two-hop or more general relay networks? These remain interesting open questions.

APPENDIX

A. Proof of Proposition 1

First, suppose that at the optimal solution to problem (73), which is denoted by β_k^* 's and λ_m^* 's, there exists a \bar{k} such that

$$\sum_{j \neq \bar{k}} \beta_j^* |\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m^* |\bar{u}_{\bar{k},m}|^2 + \sigma^2 - \frac{\beta_{\bar{k}}^* |\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_{\bar{k}}|^2}{2^{R_{\bar{k}}} - 1} > 0. \quad (160)$$

Then, consider another solution where $\lambda_m = \lambda_m^*$, $\forall m$, $\beta_k = \beta_k^*$, $\forall k \neq \bar{k}$, and

$$\beta_{\bar{k}} = \frac{\left(\sum_{j \neq \bar{k}} \beta_j^* |\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m^* |\bar{u}_{\bar{k},m}|^2 + \sigma^2 \right) (2^{R_{\bar{k}}} - 1)}{|\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_{\bar{k}}|^2} > \beta_{\bar{k}}^*. \quad (161)$$

It can be shown that the new solution is a feasible solution to problem (73). Moreover, due to (161), at the new solution, the objective value of problem (73) is increased. This contradicts to the fact that the optimal solution to problem (73) is $\{\beta_k^*, \lambda_m^*\}$. As a result, at the optimal solution to problem (73), constraint (71) should hold with equality.

Next, suppose that at the optimal solution to problem (73), which is denoted by β_k^* 's and λ_m^* 's, there exists an \bar{m} such that

$$\sum_{k=1}^K \beta_k^* |h_{k,\bar{m}}|^2 + \lambda_{\bar{m}}^* + \sigma^2 - 2^{C_{\bar{m}}} \lambda_{\bar{m}}^* > 0. \quad (162)$$

Then, consider another solution where $\beta_k = \beta_k^*$, $\forall k$, $\lambda_m = \lambda_m^*$, $\forall m \neq \bar{m}$, and

$$\lambda_{\bar{m}} = \frac{\sum_{k=1}^K \beta_k^* |h_{k,\bar{m}}|^2 + \sigma^2}{2^{C_{\bar{m}}} - 1} > \lambda_{\bar{m}}^*. \quad (163)$$

It can be shown that the new solution is a feasible solution to problem (73). Moreover, due to (163), at the new solution, the objective value of problem (73) is increased. This contradicts to the fact that the optimal solution to problem (73) is $\{\beta_k^*, \lambda_m^*\}$. As a result, at the optimal solution to problem (73), constraint (72) should hold with equality.

Proposition 1 is thus proved.

$$I_k(\boldsymbol{\beta}) = \frac{\left(\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \frac{\left(\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \sigma^2 \right) |\bar{u}_{k,m}|^2}{2^{C_m-1}} + \sigma^2 \right) (2^{R_k} - 1)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}. \quad (166)$$

$$I_k(\alpha \boldsymbol{\beta}) = \frac{\left(\sum_{j \neq k} \alpha \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \frac{\left(\sum_{k=1}^K \alpha \beta_k |h_{k,m}|^2 + \sigma^2 \right) |\bar{u}_{k,m}|^2}{2^{C_m-1}} + \sigma^2 \right) (2^{R_k} - 1)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \quad (168)$$

$$< \frac{\alpha \left(\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \frac{\left(\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \sigma^2 \right) |\bar{u}_{k,m}|^2}{2^{C_m-1}} + \sigma^2 \right) (2^{R_k} - 1)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \quad (169)$$

$$= \alpha I_k(\boldsymbol{\beta}), \quad \forall k \in \mathcal{K}. \quad (170)$$

$$I_k(\bar{\boldsymbol{\beta}}) = \frac{\left(\sum_{j \neq k} \bar{\beta}_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \frac{\left(\sum_{k=1}^K \bar{\beta}_k |h_{k,m}|^2 + \sigma^2 \right) |\bar{u}_{k,m}|^2}{2^{C_m-1}} + \sigma^2 \right) (2^{R_k} - 1)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \quad (171)$$

$$\geq \frac{\left(\sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \frac{\left(\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \sigma^2 \right) |\bar{u}_{k,m}|^2}{2^{C_m-1}} + \sigma^2 \right) (2^{R_k} - 1)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \quad (172)$$

$$= I_k(\boldsymbol{\beta}), \quad \forall k \in \mathcal{K}. \quad (173)$$

$$2^{C_m} \mathbf{Q}^{(m,m)} - \mathbf{Q}^{(m,m)} - \sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,m}|^2 - 2^{C_m} \mathbf{Q}^{(m,m+1:M)} (\mathbf{Q}^{(m+1:M,m+1:M)})^{-1} \mathbf{Q}^{(m+1:M,m)} \geq \mathbf{0}, \quad \forall m. \quad (174)$$

B. Proof of Proposition 2

First, according to (75), it follows that

$$\lambda_m = \frac{\sum_{k=1}^K \beta_k |h_{k,m}|^2 + \sigma^2}{2^{C_m} - 1}, \quad \forall m \in \mathcal{M}. \quad (164)$$

By plugging (164) into (74), it follows that

$$\beta_k = I_k(\boldsymbol{\beta}), \quad \forall k \in \mathcal{K}, \quad (165)$$

where $\boldsymbol{\beta} = [\beta_1, \dots, \beta_K]^T$, and $I_k(\boldsymbol{\beta})$ is given in (166) on top of the page. It can be shown that with $\beta_k > 0, \forall k$, we have

$$I_k(\boldsymbol{\beta}) > 0, \quad \forall k \in \mathcal{K}. \quad (167)$$

Next, given $\alpha > 1$, it can be shown that (168)–(170) on top of the page are true. Last, given $\bar{\boldsymbol{\beta}} = [\bar{\beta}_1, \dots, \bar{\beta}_K]^T \geq \boldsymbol{\beta}$, it follows that (171)–(173) on top of the page are true. As a result, the function $\mathbf{I}(\boldsymbol{\beta}) = [I_1(\boldsymbol{\beta}), \dots, I_K(\boldsymbol{\beta})]^T$ defined by (166) is a standard interference function [49]. It then follows from [49, Theorem 1] that there exists a unique solution $\boldsymbol{\beta} > \mathbf{0}$

to equation (165). Moreover, with this solution of $\boldsymbol{\beta}$, there is a unique solution of $\lambda_m > 0$'s to equation (164). As a result, if problem (78) is feasible, there exists only one solution to the constraints (74), (75), (76), and (77) in problem (78).

Proposition 2 is thus proved.

C. Proof of Proposition 4

First, it can be easily verified that the SINR constraints (104) are equivalent to the original SINR constraints (59). Next, we consider the fronthaul constraints (98), which can be re-formulated as (174) on top of the page.

Given $m = 1, \dots, M$, define \mathbf{D}_m as in (175) on top of the next page. Note that $\mathbf{Q}^{(m,m+1:M)} = (\mathbf{Q}^{(m+1:M,m+1:M)})^H$ and thus \mathbf{D}_m is a Hermitian matrix, $m = 1, \dots, M$. Note that we must have $\mathbf{Q} \succ \mathbf{0}$ in problem (97) since otherwise, the fronthaul rates for relays become infinite based on (33) and (52). According to [50, Theorem 7.7.9], given $\mathbf{Q} \succ \mathbf{0}$, (174) is equivalent to $\mathbf{D}_m \succeq \mathbf{0}, \forall m$.

$$\mathbf{D}_m = \begin{bmatrix} 2^{C_m} \mathbf{Q}^{(m,m)} - \mathbf{Q}^{(m,m)} - \sum_{k=1}^K p_k^{\text{dl}} |\bar{u}_{k,m}|^2 & 2^{C_m} \mathbf{Q}^{(m,m+1:M)} \\ 2^{C_m} \mathbf{Q}^{(m+1:M,m)} & 2^{C_m} \mathbf{Q}^{(m+1:M,m+1:M)} \end{bmatrix} \in \mathbb{C}^{(M-m+1) \times (M-m+1)}. \quad (175)$$

$$2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m) \times (m-1)} & \mathbf{Q}^{(m:M,m:M)} \end{bmatrix} - \mathbf{E}_m (\mathbf{Q} + \Psi) \mathbf{E}_m^H = \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \mathbf{D}_m \end{bmatrix}, \quad \forall m. \quad (176)$$

$$\begin{aligned} & \mathcal{L}(\{p_k^{\text{dl}}\}, \mathbf{Q}, \{\beta_k\}, \{\Lambda_m\}) \\ &= \sum_{k=1}^K p_k^{\text{dl}} \sigma^2 + \text{tr}(\mathbf{Q}) \sigma^2 - \sum_{k=1}^K \beta_k \left(\frac{p_k^{\text{dl}} |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} - \sum_{j \neq k} p_j^{\text{dl}} |\bar{\mathbf{u}}_j^H \mathbf{h}_k|^2 - \text{tr}(\mathbf{Q} \mathbf{h}_k \mathbf{h}_k^H) - \sigma^2 \right) \\ & \quad - \sum_{m=1}^M \text{tr} \left(\Lambda_m \left(2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \mathbf{Q}^{(m:M,m:M)} \end{bmatrix} - \mathbf{E}_m (\mathbf{Q} + \Psi) \mathbf{E}_m^H \right) \right) \\ &= \sum_{k=1}^K p_k^{\text{dl}} \left(\sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \right) + \sum_{k=1}^K \beta_k \sigma^2 \\ & \quad + \text{tr} \left(\mathbf{Q} \left(\sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m - \sum_{m=1}^M 2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \Lambda_m^{(m:M,m:M)} \end{bmatrix} \right) \right). \quad (177) \end{aligned}$$

Moreover, since (176) on top of the page is true, the fronthaul constraint (174) in the broadcast relay channel is equivalent to (105) given (99).

To summarize, the new SINR constraints (104) and fronthaul constraints (105) are equivalent to the original constraints. Moreover, we multiply the objective function in problem (97) by a constant σ^2 for convenience. As a result, problem (103) and problem (97) have the same optimal solution. Proposition 4 is proved.

D. Proof of Proposition 5

We can write down the Lagrangian of problem (103) as in (177) on top of the page, where $\beta_k \geq 0$, $k = 1, \dots, K$, and $\Lambda_m \in \mathbb{C}^{M \times M} \succeq \mathbf{0}$, $m = 1, \dots, M$, are the dual variables associated with constraints (104) and (105), respectively. The dual function of problem (97) is thus formulated as

$$\begin{aligned} & g(\{\beta_k\}, \{\Lambda_m\}) \\ &= \min_{p_k^{\text{dl}} \geq 0, \forall k \in \mathcal{K}, \mathbf{Q} \succeq \mathbf{0}} \mathcal{L}(\{p_k^{\text{dl}}\}, \mathbf{Q}, \{\beta_k\}, \{\Lambda_m\}). \quad (178) \end{aligned}$$

At last, the dual problem of problem (103) is

$$\text{maximize}_{\{\beta_k\}, \{\Lambda_m\}} g(\{\beta_k\}, \{\Lambda_m\}) \quad (179)$$

$$\text{subject to } \beta_k \geq 0, \forall k, \quad (180)$$

$$\Lambda_m \succeq \mathbf{0}, \forall m. \quad (181)$$

Note that according to (177), the dual function is $g(\{\beta_k\}, \{\Lambda_m\}) = \sum_{k=1}^K \beta_k \sigma^2$ if and only if

$$\begin{aligned} & \sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 \\ & \quad - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} \geq 0, \quad \forall k \quad (182) \end{aligned}$$

and

$$\begin{aligned} & \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \Lambda_m \mathbf{E}_m \\ & \quad - \sum_{m=1}^M 2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \Lambda_m^{(m:M,m:M)} \end{bmatrix} \succeq \mathbf{0}. \quad (183) \end{aligned}$$

Otherwise, the dual function is unbounded, i.e., $g(\{\beta_k\}, \{\Lambda_m\}) = -\infty$. As a result, the optimal solution to problem (178) must satisfy constraints (182) and (183). In this case, $g(\{\beta_k\}, \{\Lambda_m\}) = \sum_{k=1}^K \beta_k \sigma^2$. This indicates that the dual problem of problem (103), i.e., problem (179), is equivalent to problem (106).

Proposition 4 is thus proved.

E. Proof of Proposition 6

According to (113) and (114), we have

$$\begin{aligned}
\mathbf{A}_1 &= 2^{C_1} \mathbf{\Lambda}_1^{(1:M,1:M)} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{A}_2 \end{bmatrix} \quad (184) \\
&= 2^{C_1} \mathbf{\Lambda}_1^{(1:M,1:M)} + 2^{C_2} \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{\Lambda}_m^{(M-1:M-1)} \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times (M-2)} \\ \mathbf{0}_{(M-2) \times 2} & \mathbf{A}_3 \end{bmatrix} \quad (185) \\
&\vdots \\
&= \sum_{m=1}^M 2^{C_m} \begin{bmatrix} \mathbf{0}_{(m-1) \times (m-1)} & \mathbf{0}_{(m-1) \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times (m-1)} & \mathbf{\Lambda}_m^{(M-m+1:M-m+1)} \end{bmatrix}. \quad (186)
\end{aligned}$$

By combining (112) and (186), we have (108). As a result, problem (106) is equivalent to problem (111). Proposition 6 is thus proved.

F. Proof of Proposition 7

We first show that at the optimal solution to problem (111), condition (115) is true, i.e.,

$$\mathbf{B} = \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \mathbf{\Lambda}_m \mathbf{E}_m - \mathbf{A}_1 = \mathbf{0}. \quad (187)$$

Suppose that at the optimal solution $\{\beta_k, \mathbf{A}_m, \mathbf{\Lambda}_m\}$ to problem (111), $\mathbf{B} \succeq \mathbf{0}$ but $\mathbf{B} \neq \mathbf{0}$. Then, let \bar{m} denote the index of the first positive diagonal element of \mathbf{B} . Since $\mathbf{B} \succeq \mathbf{0}$, we have

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{(\bar{m}-1) \times (\bar{m}-1)} & \mathbf{0}_{(\bar{m}-1) \times (M-\bar{m}+1)} \\ \mathbf{0}_{(M-\bar{m}+1) \times (\bar{m}-1)} & \mathbf{B}^{(\bar{m}:M, \bar{m}:M)} \end{bmatrix}. \quad (188)$$

Consider another solution given by

$$\hat{\beta}_k = \beta_k, \quad \forall k, \quad (189)$$

$$\hat{\mathbf{\Lambda}}_m = \mathbf{\Lambda}_m, \quad \forall m \neq \bar{m}, \quad (190)$$

$$\hat{\mathbf{\Lambda}}_{\bar{m}} = \mathbf{\Lambda}_{\bar{m}} + \begin{bmatrix} \mathbf{0}_{(\bar{m}-1) \times (\bar{m}-1)} & \mathbf{0}_{(\bar{m}-1) \times (M-\bar{m}+1)} \\ \mathbf{0}_{(M-\bar{m}+1) \times (\bar{m}-1)} & \mathbf{C}_{\bar{m}} \end{bmatrix}, \quad (191)$$

$$\hat{\mathbf{A}}_m = 2^{C_m} \hat{\mathbf{\Lambda}}_m^{(m:M, m:M)} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-m)} \\ \mathbf{0}_{(M-m) \times 1} & \hat{\mathbf{A}}_{m+1} \end{bmatrix}, \quad (192)$$

$m = 1, \dots, M-1,$

$$\hat{\mathbf{A}}_M = 2^{C_M} \hat{\mathbf{\Lambda}}_M^{(M,M)}, \quad (193)$$

where $\mathbf{C}_{\bar{m}} \in \mathbb{C}^{(M-\bar{m}+1) \times (M-\bar{m}+1)}$ in (191) satisfies

$$\begin{aligned}
2^{C_{\bar{m}}} \mathbf{C}_{\bar{m}} - \begin{bmatrix} \mathbf{C}_{\bar{m}}^{(1,1)} & \mathbf{0}_{1 \times (M-\bar{m})} \\ \mathbf{0}_{(M-\bar{m}) \times 1} & \mathbf{0}_{(M-\bar{m}) \times (M-\bar{m})} \end{bmatrix} \\
= \mathbf{B}^{(\bar{m}:M, \bar{m}:M)}. \quad (194)
\end{aligned}$$

Since $\mathbf{B}^{(\bar{m}, \bar{m})} > 0$, it follows from (194) that $\mathbf{C}_{\bar{m}}^{(1,1)} > 0$. Moreover, since $\mathbf{B}^{(\bar{m}:M, \bar{m}:M)} \succeq \mathbf{0}$ and $\mathbf{C}_{\bar{m}}^{(1,1)} > 0$, according to (194) we have $\mathbf{C}_{\bar{m}} \succeq \mathbf{0}$. As a result, (191) indicates that

$\hat{\mathbf{\Lambda}}_{\bar{m}} \succeq \mathbf{0}$, i.e., constraint (110) is satisfied. Further, it follows from (192) and (193) that

$$\begin{aligned}
\hat{\mathbf{A}}_1 &= \mathbf{A}_1 + 2^{C_{\bar{m}}} \begin{bmatrix} \mathbf{0}_{(\bar{m}-1) \times (\bar{m}-1)} & \mathbf{0}_{(\bar{m}-1) \times (M-\bar{m}+1)} \\ \mathbf{0}_{(M-\bar{m}+1) \times (\bar{m}-1)} & \mathbf{C}_{\bar{m}} \end{bmatrix} \\
&= \mathbf{A}_1 + \mathbf{B} + \text{diag}([\mathbf{0}_{1 \times (\bar{m}-1)}, \mathbf{C}_{\bar{m}}^{(1,1)}, \mathbf{0}_{1 \times (M-\bar{m})}]) \\
&= \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \mathbf{\Lambda}_m \mathbf{E}_m \\
&\quad + \text{diag}([\mathbf{0}_{1 \times (\bar{m}-1)}, \mathbf{C}_{\bar{m}}^{(1,1)}, \mathbf{0}_{1 \times (M-\bar{m})}]) \\
&= \sigma^2 \mathbf{I} + \sum_{k=1}^K \hat{\beta}_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m^H \hat{\mathbf{\Lambda}}_m \mathbf{E}_m. \quad (195)
\end{aligned}$$

As a result, at the new solution, constraint (112) is satisfied with equality. Last, since $\mathbf{C}_{\bar{m}}^{(1,1)} > 0$, it follows from (190) and (191) that $\hat{\mathbf{\Lambda}}_m^{(m,m)} = \mathbf{\Lambda}_m^{(m,m)}$, $\forall m \neq \bar{m}$, and $\hat{\mathbf{\Lambda}}_{\bar{m}}^{(\bar{m}, \bar{m})} > \mathbf{\Lambda}_{\bar{m}}^{(\bar{m}, \bar{m})}$. Moreover, according to condition (52), we have $\sum_{k=1}^K |\bar{u}_{k, \bar{m}}|^2 > 0$. As a result, at the new solution, constraint (107) is satisfied with strict inequality at least for one k .

To summarize, at the new solution, the same objective value of problem (111) is achieved due to (189), and all the constraints are also satisfied. Moreover, since constraint (107) is satisfied with inequality, we can further find a better solution of β_k 's such that the objective value of problem (111) is enhanced while constraint (107) is satisfied with equality. This contradicts the assumption that the optimal solution to problem (111) is $\{\beta_k, \mathbf{A}_m, \mathbf{\Lambda}_m\}$. As a result, at the optimal solution to problem (111), constraint (112) must hold with equality.

Next, we show that at the optimal solution to problem (111), conditions (116) and (117) are true.

First, according to (115), at the optimal solution, we have $\mathbf{A}_1 \succ \mathbf{0}$, and thus

$$\mathbf{A}_1^{(1,1)} > 0. \quad (196)$$

Then, we show that at the optimal solution, condition (117) is true for $m = 1$, i.e.,

$$2^{C_1} \mathbf{\Lambda}_1 = \frac{\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}}{\mathbf{A}_1^{(1,1)}}. \quad (197)$$

On the right-hand side (RHS) of (113), given each m , the first row and the first column of the overall matrix is merely contributed by the first row and the first column of $\mathbf{\Lambda}_m^{(m:M, m:M)}$. With $\mathbf{A}_1^{(1,1)} > 0$, to make (113) hold, the optimal solution $2^{C_1} \mathbf{\Lambda}_1$ must be in the following form

$$2^{C_1} \mathbf{\Lambda}_1 = \frac{\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}}{\mathbf{A}_1^{(1,1)}} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{T}_1 \end{bmatrix}, \quad (198)$$

where \mathbf{T}_1 satisfies

$$\begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{T}_1 + \mathbf{A}_2 \end{bmatrix} = \mathbf{A}_1 - \frac{\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}}{\mathbf{A}_1^{(1,1)}}. \quad (199)$$

It can be shown from (199) that $\mathbf{T}_1 + \mathbf{A}_2 \succeq \mathbf{0}$.

$$\tilde{\mathbf{A}}_m = \begin{cases} \mathbf{A}_m, & \text{if } m = 1 \text{ or } 1 + \tilde{m} < m \leq M, \\ \mathbf{A}_m + \begin{bmatrix} \mathbf{0}_{(1+\tilde{m}-m) \times (1+\tilde{m}-m)} & \mathbf{0}_{(1+\tilde{m}-m) \times (M-\tilde{m})} \\ \mathbf{0}_{(M-\tilde{m}) \times (1+\tilde{m}-m)} & \mathbf{T}_1^{(\tilde{m}:M-1, \tilde{m}:M-1)} / 2^{C_1} \end{bmatrix}, & \text{if } 1 < m < 1 + \tilde{m}, \\ \mathbf{A}_m + \mathbf{T}_1^{(\tilde{m}:M-1, \tilde{m}:M-1)} / 2^{C_1}, & \text{if } m = 1 + \tilde{m}. \end{cases} \quad (207)$$

Lemma 3: Suppose that $\mathbf{a} \in \mathbb{C}^{M \times 1}$ with $\mathbf{a}^{(1)} > 0$ and $\mathbf{B} \in \mathbb{C}^{(M-1) \times (M-1)}$. Then,

$$\mathbf{a}\mathbf{a}^H + \begin{bmatrix} 0 & \mathbf{0}_{1 \times M} \\ \mathbf{0}_{M \times 1} & \mathbf{B} \end{bmatrix} \succeq \mathbf{0}, \quad (200)$$

implies $\mathbf{B} \succeq \mathbf{0}$.

Proof: Suppose that \mathbf{B} is not a positive semidefinite matrix. Then, there exists some $\mathbf{x} \in \mathbb{C}^{(M-1) \times 1}$ such that $\mathbf{x}^H \mathbf{B} \mathbf{x} < 0$. Next, with $\mathbf{a}^{(1)} > 0$, define $\mathbf{y} = [-\mathbf{x}^H \mathbf{a}^{(2:M)} / \mathbf{a}^{(1)}, \mathbf{x}^H]^H \in \mathbb{C}^{M \times 1}$. It can then be shown that

$$\mathbf{y}^H \left(\mathbf{a}\mathbf{a}^H + \begin{bmatrix} 0 & \mathbf{0}_{1 \times M} \\ \mathbf{0}_{M \times 1} & \mathbf{B} \end{bmatrix} \right) \mathbf{y} = \mathbf{x}^H \mathbf{B} \mathbf{x} < 0. \quad (201)$$

As a result, as long as \mathbf{B} is not a positive semidefinite matrix, (200) does not hold. Lemma 3 is thus proved. ■

According to Lemma 3 and (196), if $2^{C_1} \mathbf{\Lambda}_1$ defined in (198) is a positive semidefinite matrix, we must have

$$\mathbf{T}_1 \succeq \mathbf{0}. \quad (202)$$

Now, suppose that the optimal solution $\{\beta_k, \mathbf{\Lambda}_m, \mathbf{A}_m\}$ to problem (111) satisfies $\mathbf{T}_1 \neq \mathbf{0}$. Define \tilde{m} as the index of the first positive diagonal element in \mathbf{T}_1 . Since $\mathbf{T}_1 \succeq \mathbf{0}$, \mathbf{T}_1 must be in the form of

$$\mathbf{T}_1 = \begin{bmatrix} \mathbf{0}_{(\tilde{m}-1) \times (\tilde{m}-1)} & \mathbf{0}_{(\tilde{m}-1) \times (M-\tilde{m})} \\ \mathbf{0}_{(M-\tilde{m}) \times (\tilde{m}-1)} & \mathbf{T}_1^{(\tilde{m}:M-1, \tilde{m}:M-1)} \end{bmatrix}, \quad (203)$$

with $\mathbf{T}_1^{(\tilde{m}:M-1, \tilde{m}:M-1)} \succeq \mathbf{0}$.

Then, consider a new solution in which $\tilde{\beta}_k = \beta_k, \forall k, \tilde{\mathbf{\Lambda}}_m$'s are given as follows

$$\begin{aligned} \tilde{\mathbf{\Lambda}}_1 &= \mathbf{\Lambda}_1 - \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{T}_1 / 2^{C_1} \end{bmatrix} \\ &= \frac{\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}}{\mathbf{A}_1^{(1,1)}} \\ &\succeq \mathbf{0}, \end{aligned} \quad (204)$$

$$\begin{aligned} \tilde{\mathbf{\Lambda}}_{1+\tilde{m}} &= \mathbf{\Lambda}_{1+\tilde{m}} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{T}_1 / 2^{C_1} \end{bmatrix} \\ &= \mathbf{\Lambda}_{1+\tilde{m}} + \begin{bmatrix} \mathbf{0}_{\tilde{m} \times \tilde{m}} & \mathbf{0}_{\tilde{m} \times (M-\tilde{m})} \\ \mathbf{0}_{(M-\tilde{m}) \times \tilde{m}} & \mathbf{T}_1^{(\tilde{m}:M-1, \tilde{m}:M-1)} / 2^{C_1} \end{bmatrix} \\ &\succeq \mathbf{0}, \end{aligned} \quad (205)$$

$$\tilde{\mathbf{\Lambda}}_m = \mathbf{\Lambda}_m, \quad \forall m \neq 1 \text{ and } m \neq 1 + \tilde{m}, \quad (206)$$

and $\tilde{\mathbf{A}}_m$'s are given as in (207) on top of the page.

It can be shown that the above solution satisfies constraints (113), (114), (109), and (110) in problem (111). Moreover, with this new solution, we have

$$\tilde{\mathbf{\Lambda}}_1^{(1,1)} = \mathbf{\Lambda}_1^{(1,1)} - 0 = \mathbf{\Lambda}_1^{(1,1)}, \quad (208)$$

$$\tilde{\mathbf{\Lambda}}_{1+\tilde{m}}^{(1+\tilde{m}, 1+\tilde{m})} = \mathbf{\Lambda}_{1+\tilde{m}}^{(1+\tilde{m}, 1+\tilde{m})} + \mathbf{T}_1^{(\tilde{m}, \tilde{m})} / 2^{C_1}, \quad (209)$$

$$\tilde{\mathbf{\Lambda}}_m^{(m,m)} = \mathbf{\Lambda}_m^{(m,m)}, \quad \forall m \neq 1 \text{ and } m \neq 1 + \tilde{m}. \quad (210)$$

Since $\mathbf{T}_1^{(\tilde{m}, \tilde{m})} > 0$, it follows that $\tilde{\mathbf{\Lambda}}_m^{(m,m)} = \mathbf{\Lambda}_m^{(m,m)}$ if $m \neq 1 + \tilde{m}$, and $\tilde{\mathbf{\Lambda}}_m^{(m,m)} > \mathbf{\Lambda}_m^{(m,m)}$ if $m = 1 + \tilde{m}$. Moreover, according to (52), we have $\sum_{k=1}^K |\tilde{u}_{k, 1+\tilde{m}}|^2 > 0$. As a result, at the new solution, constraint (107) in problem (111) is satisfied with strict inequality at least for one k . Last, since

$$\sum_{m=1}^M \mathbf{E}_m^H \tilde{\mathbf{\Lambda}}_m \mathbf{E}_m \succeq \sum_{m=1}^M \mathbf{E}_m^H \mathbf{\Lambda}_m \mathbf{E}_m, \quad (211)$$

constraint (112) in problem (111) is satisfied.

To summarize, with the new solution $\{\tilde{\beta}_k, \tilde{\mathbf{\Lambda}}_m, \tilde{\mathbf{A}}_m\}$, all the constraints in problem (111) are satisfied, while the optimal objective value is also achieved. Moreover, since constraint (107) is satisfied with strict inequality for some k 's, we can further find a better solution of β_k 's such that the objective value is enhanced while constraint (107) is satisfied with equality. This contradicts the assumption that the optimal solution to problem (111) is $\{\beta_k, \mathbf{\Lambda}_m, \mathbf{A}_m\}$. As a result, at the optimal solution to problem (111), condition (197) must hold.

Given (197), it follows from (113) that

$$\begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-1)} \\ \mathbf{0}_{(M-1) \times 1} & \mathbf{A}_2 \end{bmatrix} = \mathbf{A}_1 - \frac{\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}}{\mathbf{A}_1^{(1,1)}}. \quad (212)$$

Since $\text{rank}(\mathbf{A}_1) = M$ and $\text{rank}(\mathbf{A}_1^{(1:M,1)} \mathbf{A}_1^{(1,1:M)}) = 1$, we have $\text{rank}(\mathbf{A}_2) = M - 1$. As a result, we have $\mathbf{A}_2 \succ \mathbf{0}$, and thus $\mathbf{A}_2^{(1,1)} > 0$. Similar to the way to prove (197), we can show that with $\mathbf{A}_2^{(1,1)} > 0$, condition (117) must hold for $m = 2$. Then, similar to the way to prove $\mathbf{A}_2 \succ \mathbf{0}$, we can show that if condition (117) holds for $m = 2$, then $\mathbf{A}_3 \succ \mathbf{0}$. We can keep applying the above method to show that with the optimal solution to problem (111), conditions (116) and (117) are true.

Proposition 7 is thus proved.

G. Proof of Proposition 8

According to (113), (114) and (117), the optimal solution to problem (111) must satisfy

$$\begin{aligned} &\begin{bmatrix} 0 & \mathbf{0}_{1 \times (M-m+1)} \\ \mathbf{0}_{(M-m+1) \times 1} & \mathbf{A}_m \end{bmatrix} \\ &= \mathbf{A}_{m-1} - 2^{C_{m-1}} \mathbf{\Lambda}_{m-1}^{(m-1:M, m-1:M)} \\ &= \mathbf{A}_{m-1} - \frac{\mathbf{A}_{m-1}^{(1:M-m+2,1)} \mathbf{A}_{m-1}^{(1,1:M-m+2)}}{\mathbf{A}_{m-1}^{(1,1)}}, \quad m = 2, \dots, M. \end{aligned} \quad (213)$$

$$\begin{aligned} & \mathbf{A}_n^{(m-n+1, m-n+1)} - \mathbf{A}_n^{(m-n+1, 1: m-n)} (\mathbf{A}_n^{(1: m-n, 1: m-n)})^{-1} \mathbf{A}_n^{(1: m-n, m-n+1)} \\ &= \mathbf{A}_{n-1}^{(m-n+2, m-n+2)} - \mathbf{A}_{n-1}^{(m-n+2, 1: m-n+1)} (\mathbf{A}_{n-1}^{(1: m-n+1, 1: m-n+1)})^{-1} \mathbf{A}_{n-1}^{(1: m-n+1, m-n+2)}, \quad n = 2, \dots, m. \end{aligned} \quad (217)$$

$$\begin{aligned} & \mathbf{A}_n^{(m-n+1, m-n+1)} - \mathbf{A}_n^{(m-n+1, 1: m-n)} (\mathbf{A}_n^{(1: m-n, 1: m-n)})^{-1} \mathbf{A}_n^{(1: m-n, m-n+1)} \\ &= \mathbf{A}_{n-1}^{(m-n+2, m-n+2)} - \frac{\mathbf{A}_{n-1}^{(m-n+2, 1)} \mathbf{A}_{n-1}^{(1, m-n+2)}}{\mathbf{A}_{n-1}^{(1, 1)}} - \left(\mathbf{A}_{n-1}^{(m-n+2, 2: m-n+1)} - \frac{\mathbf{A}_{n-1}^{(m-n+2, 1)} \mathbf{A}_{n-1}^{(1, 2: m-n+1)}}{\mathbf{A}_{n-1}^{(1, 1)}} \right) \\ & \times \mathbf{B}_{n-1, m}^{-1} \left(\mathbf{A}_{n-1}^{(2: m-n+1, m-n+2)} - \frac{\mathbf{A}_{n-1}^{(1, m-n+2)} \mathbf{A}_{n-1}^{(2: m-n+1, 1)}}{\mathbf{A}_{n-1}^{(1, 1)}} \right), \quad n = 2, \dots, m. \end{aligned} \quad (218)$$

$$\begin{aligned} & \mathbf{A}_{n-1}^{(m-n+2, m-n+2)} - \mathbf{A}_{n-1}^{(m-n+2, 1: m-n+1)} \times (\mathbf{A}_{n-1}^{(1: m-n+1, 1: m-n+1)})^{-1} \mathbf{A}_{n-1}^{(1: m-n+1, m-n+2)} \\ &= \mathbf{A}_{n-1}^{(m-n+2, m-n+2)} - [\mathbf{A}_{n-1}^{(m-n+2, 1)}, \mathbf{A}_{n-1}^{(m-n+2, 2: m-n+1)}] \\ & \times \begin{pmatrix} \mathbf{A}_{n-1}^{(1, 1)} & \mathbf{A}_{n-1}^{(1, 2: m-n+1)} \\ \mathbf{A}_{n-1}^{(2: m-n+1, 1)} & \mathbf{A}_{n-1}^{(2: m-n+1, 2: m-n+1)} \end{pmatrix}^{-1} \times \begin{bmatrix} \mathbf{A}_{n-1}^{(1, m-n+2)} \\ \mathbf{A}_{n-1}^{(2: m-n+1, m-n+2)} \end{bmatrix}, \quad n = 2, \dots, m. \end{aligned} \quad (220)$$

In other words, we have

$$\begin{aligned} \mathbf{A}_m &= \mathbf{A}_{m-1}^{(2: M-m+2, 2: M-m+2)} \\ & - \frac{\mathbf{A}_{m-1}^{(2: M-m+2, 1)} \mathbf{A}_{m-1}^{(1, 2: M-m+2)}}{\mathbf{A}_{m-1}^{(1, 1)}}, \quad m = 2, \dots, M. \end{aligned} \quad (215)$$

It then follows from (215) that

$$\mathbf{A}_m^{(1, 1)} = \mathbf{A}_{m-1}^{(2, 2)} - \frac{\mathbf{A}_{m-1}^{(2, 1)} \mathbf{A}_{m-1}^{(1, 2)}}{\mathbf{A}_{m-1}^{(1, 1)}}, \quad m = 2, \dots, M. \quad (216)$$

Lemma 4: If (215) holds and $\mathbf{A}_m \succ \mathbf{0}$, $\forall m$, then for any $m = 2, \dots, M$, we have (217) on top of the page.

Proof: First, if (215) holds, (218) on top of the page then follows, where

$$\begin{aligned} \mathbf{B}_{n-1, m} &= \mathbf{A}_{n-1}^{(2: m-n+1, 2: m-n+1)} \\ & - \frac{\mathbf{A}_{n-1}^{(2: m-n+1, 1)} \mathbf{A}_{n-1}^{(1, 2: m-n+1)}}{\mathbf{A}_{n-1}^{(1, 1)}}. \end{aligned} \quad (219)$$

On the other hand, we have (220) on top of the page.

Lemma 5: [50, Section 0.7.3] Consider an invertible matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{bmatrix}. \quad (221)$$

Then the inverse of \mathbf{X} is given in (222) on top of the next page.

According to Lemma 5, we have (223) on the next page. By taking (223) into (220), (224) on the next page follows. According to (218) and (224), it is concluded that if (215) is true, then (217) holds. Lemma 4 is thus proved. ■

By combining (216), Proposition 7, and Lemma 4, the optimal solution to problem (111) satisfies

$$\begin{aligned} & \mathbf{A}_m^{(1, 1)} \\ &= \mathbf{A}_{m-1}^{(2, 2)} - \frac{\mathbf{A}_{m-1}^{(2, 1)} \mathbf{A}_{m-1}^{(1, 2)}}{\mathbf{A}_{m-1}^{(1, 1)}} \end{aligned} \quad (225)$$

$$= \mathbf{A}_{m-2}^{(3, 3)} - \mathbf{A}_{m-2}^{(3, 1: 2)} (\mathbf{A}_{m-2}^{(1: 2, 1: 2)})^{-1} \mathbf{A}_{m-2}^{(1: 2, 3)} \quad (226)$$

⋮

$$= \mathbf{A}_1^{(m, m)} - \mathbf{A}_1^{(m, 1: m-1)} (\mathbf{A}_1^{(1: m-1, 1: m-1)})^{-1} \mathbf{A}_1^{(1: m-1, m)}, \quad m = 2, \dots, M. \quad (227)$$

Moreover, according to (117), it follows that

$$2^C \mathbf{\Lambda}_m^{(m, m)} = \mathbf{A}_m^{(1, 1)}, \quad \forall m. \quad (228)$$

By combining (227) and (228), it can be concluded that the optimal solution to problem (111) must satisfy (229) on the next page. To summarize, according to constraints (113) and (114) in problem (111), the optimal $\mathbf{\Lambda}_m^{(m, m)}$'s are just functions of \mathbf{A}_1 as shown in (229). Moreover, constraints (112) and (107) are only dependent of $\mathbf{\Lambda}_m^{(m, m)}$'s, but are independent of the other elements of $\mathbf{\Lambda}_m$'s. As a result, problem (111) is equivalent to the following problem

$$\begin{aligned} & \text{maximize} && \sum_{k=1}^K \beta_k \sigma^2 \\ & \{\beta_k\}, \{\mathbf{\Lambda}_m^{(m, m)}\}, \mathbf{A}_1 \end{aligned} \quad (230)$$

$$\text{subject to} \quad \mathbf{A}_1 \succ \mathbf{0}, \quad (231)$$

$$(107), (109), (112), (121), (229).$$

As compared to problem (111), the optimization variables $\mathbf{A}_2, \dots, \mathbf{A}_M$ disappear in problem (230). Moreover, constraints (113) and (114) reduce to constraint (229).

$$\mathbf{X}^{-1} = \begin{bmatrix} \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{D}(\mathbf{F} - \mathbf{E}\mathbf{C}^{-1}\mathbf{D})^{-1}\mathbf{E}\mathbf{C}^{-1} & -\mathbf{C}^{-1}\mathbf{D}(\mathbf{F} - \mathbf{E}\mathbf{C}^{-1}\mathbf{D})^{-1} \\ -(\mathbf{F} - \mathbf{E}\mathbf{C}^{-1}\mathbf{D})^{-1}\mathbf{E}\mathbf{C}^{-1} & (\mathbf{F} - \mathbf{E}\mathbf{C}^{-1}\mathbf{D})^{-1} \end{bmatrix}. \quad (222)$$

$$\begin{aligned} & [\mathbf{A}_{n-1}^{(m-n+2,1)}, \mathbf{A}_{n-1}^{(m-n+2,2:m-n+1)}] \left(\begin{array}{cc} \mathbf{A}_{n-1}^{(1,1)} & \mathbf{A}_{n-1}^{(1,2:m-n+1)} \\ \mathbf{A}_{n-1}^{(2:m-n+1,1)} & \mathbf{A}_{n-1}^{(2:m-n+1,2:m-n+1)} \end{array} \right)^{-1} \begin{bmatrix} \mathbf{A}_{n-1}^{(1,m-n+2)} \\ \mathbf{A}_{n-1}^{(2:m-n+1,m-n+2)} \end{bmatrix} \\ &= [\mathbf{A}_{n-1}^{(m-n+2,1)}, \mathbf{A}_{n-1}^{(m-n+2,2:m-n+1)}] \\ & \quad \times \begin{bmatrix} \frac{1}{\mathbf{A}_{n-1}^{(1,1)}} - \frac{\mathbf{A}_{n-1}^{(1,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,1)}}{(\mathbf{A}_{n-1}^{(1,1)})^2} & -\frac{\mathbf{A}_{n-1}^{(1,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}}{\mathbf{A}_{n-1}^{(1,1)}} \\ -\frac{\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,1)}}{\mathbf{A}_{n-1}^{(1,1)}} & \mathbf{B}_{n-1,m}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{n-1}^{(1,m-n+2)} \\ \mathbf{A}_{n-1}^{(2:m-n+1,m-n+2)} \end{bmatrix} \\ &= \frac{\mathbf{A}_{n-1}^{(m-n+2,1)}\mathbf{A}_{n-1}^{(1,m-n+2)}}{\mathbf{A}_{n-1}^{(1,1)}} + \frac{\mathbf{A}_{n-1}^{(m-n+2,1)}\mathbf{A}_{n-1}^{(1,m-n+2)}\mathbf{A}_{n-1}^{(1,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,1)}}{(\mathbf{A}_{n-1}^{(1,1)})^2} \\ & \quad - \frac{\mathbf{A}_{n-1}^{(1,m-n+2)}\mathbf{A}_{n-1}^{(m-n+2,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,1)}}{\mathbf{A}_{n-1}^{(1,1)}} - \frac{\mathbf{A}_{n-1}^{(m-n+2,1)}\mathbf{A}_{n-1}^{(1,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,m-n+2)}}{\mathbf{A}_{n-1}^{(1,1)}} \\ & \quad + \mathbf{A}_{n-1}^{(m-n+2,2:m-n+1)}\mathbf{B}_{n-1,m}^{-1}\mathbf{A}_{n-1}^{(2:m-n+1,m-n+2)}, \quad n = 2, \dots, m-1. \end{aligned} \quad (223)$$

$$\begin{aligned} & \mathbf{A}_{n-1}^{(m-n+2,m-n+2)} - \mathbf{A}_{n-1}^{(m-n+2,1:m-n+1)} \times (\mathbf{A}_{n-1}^{(1:m-n+1,1:m-n+1)})^{-1} \mathbf{A}_{n-1}^{(1:m-n+1,m-n+2)} \\ &= \mathbf{A}_{n-1}^{(m-n+2,m-n+2)} - \frac{\mathbf{A}_{n-1}^{(m-n+2,1)}\mathbf{A}_{n-1}^{(1,m-n+2)}}{\mathbf{A}_{n-1}^{(1,1)}} - \left(\mathbf{A}_{n-1}^{(m-n+2,2:m-n+1)} - \frac{\mathbf{A}_{n-1}^{(m-n+2,1)}\mathbf{A}_{n-1}^{(1,2:m-n+1)}}{\mathbf{A}_{n-1}^{(1,1)}} \right) \\ & \quad \times \mathbf{B}_{n-1,m}^{-1} \left(\mathbf{A}_{n-1}^{(2:m-n+1,m-n+2)} - \frac{\mathbf{A}_{n-1}^{(1,m-n+2)}\mathbf{A}_{n-1}^{(2:m-n+1,1)}}{\mathbf{A}_{n-1}^{(1,1)}} \right), \quad n = 2, \dots, m. \end{aligned} \quad (224)$$

$$2^C \mathbf{\Lambda}_m^{(m,m)} = \begin{cases} \mathbf{A}_1^{(1,1)} & \text{if } m = 1, \\ \mathbf{A}_1^{(m,m)} - \mathbf{A}_1^{(m,1:m-1)} (\mathbf{A}_1^{(1:m-1,1:m-1)})^{-1} \mathbf{A}_1^{(1:m-1,m)}, & \text{if } m = 2, \dots, M. \end{cases} \quad (229)$$

Lemma 6: Consider $\mathbf{X} \in \mathbb{C}^{M \times M}$ and $\mathbf{Y} \in \mathbb{C}^{M \times M}$. If $\mathbf{X} \succ \mathbf{0}$, $\mathbf{Y} \succ \mathbf{0}$, and $\mathbf{X} \succeq \mathbf{Y}$, it then follows that

$$\mathbf{X}^{(1,1)} \geq \mathbf{Y}^{(1,1)}, \quad (232)$$

and

$$\begin{aligned} & \mathbf{X}^{(m,m)} - \mathbf{X}^{(m,1:m-1)} (\mathbf{X}^{(1:m-1,1:m-1)})^{-1} \mathbf{X}^{(1:m-1,m)} \\ & \geq \mathbf{Y}^{(m,m)} - \mathbf{Y}^{(m,1:m-1)} (\mathbf{Y}^{(1:m-1,1:m-1)})^{-1} \mathbf{Y}^{(1:m-1,m)}, \\ & \quad m = 2, \dots, M. \end{aligned} \quad (233)$$

Proof: (232) directly follows from $\mathbf{X} \succeq \mathbf{Y}$. In the following, we prove (233). Since $\mathbf{X} \succ \mathbf{0}$, $\mathbf{Y} \succ \mathbf{0}$, and $\mathbf{Y} \succeq \mathbf{X}$, we have $\mathbf{X}^{(1:m,1:m)} \succ \mathbf{0}$, $\mathbf{Y}^{(1:m,1:m)} \succ \mathbf{0}$, and $\mathbf{X}^{(1:m,1:m)} \succeq \mathbf{Y}^{(1:m,1:m)}$, $\forall m$. As a result, the inverses of $\mathbf{X}^{(1:m-1,1:m-1)}$'s and $\mathbf{Y}^{(1:m-1,1:m-1)}$'s exist, $\forall m \geq 2$, and (234)–(237) on top of the next page are true.

Lemma 6 is thus proved. \blacksquare

According to Lemma 6, (112) and (229) indicate that

$$\begin{aligned} & \mathbf{\Omega}^{(m,m)} - \mathbf{\Omega}^{(m,1:m-1)} (\mathbf{\Omega}^{(1:m-1,1:m-1)})^{-1} \mathbf{\Omega}^{(1:m-1,m)} \\ & \geq \mathbf{A}_1^{(m,m)} - \mathbf{A}_1^{(m,1:m-1)} (\mathbf{A}_1^{(1:m-1,1:m-1)})^{-1} \mathbf{A}_1^{(1:m-1,m)} \\ & = 2^C \mathbf{\Lambda}_m^{(m,m)}, \quad m = 1, \dots, M. \end{aligned} \quad (238)$$

This corresponds to constraint (120). As a result, any feasible solution to problem (230) is feasible to problem (119). In other words, the optimal value of problem (119) is no smaller than that of problem (230).

Next, we show that the optimal value of problem (119) is no larger than that of problem (230).

Lemma 7: The optimal solution to problem (119) must satisfy

$$\begin{aligned} & \mathbf{\Omega}^{(m,m)} - \mathbf{\Omega}^{(m,1:m-1)} (\mathbf{\Omega}^{(1:m-1,1:m-1)})^{-1} \mathbf{\Omega}^{(1:m-1,m)} \\ & = 2^C \mathbf{\Lambda}_m^{(m,m)}, \quad m = 1, \dots, M. \end{aligned} \quad (239)$$

$$\begin{aligned} & \mathbf{X}^{(m,m)} - \mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1}\mathbf{X}^{(1:m-1,m)} \\ &= [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1}, \mathbf{1}] \times \mathbf{X}^{(1:m,1:m)} \times [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1}, \mathbf{1}]^H \end{aligned} \quad (234)$$

$$\geq [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1}, \mathbf{1}] \times \mathbf{Y}^{(1:m,1:m)} \times [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1}, \mathbf{1}]^H \quad (235)$$

$$\begin{aligned} &= \mathbf{Y}^{(m,m)} - \mathbf{Y}^{(m,1:m-1)}(\mathbf{Y}^{(1:m-1,1:m-1)})^{-1}\mathbf{Y}^{(1:m-1,m)} + [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1} \\ &\quad + \mathbf{Y}^{(m,1:m-1)}(\mathbf{Y}^{(1:m-1,1:m-1)})^{-1}] \times \mathbf{Y}^{(1:m-1,1:m-1)} \times [-\mathbf{X}^{(m,1:m-1)}(\mathbf{X}^{(1:m-1,1:m-1)})^{-1} \\ &\quad + \mathbf{Y}^{(m,1:m-1)}(\mathbf{Y}^{(1:m-1,1:m-1)})^{-1}]^H \end{aligned} \quad (236)$$

$$\geq \mathbf{Y}^{(m,m)} - \mathbf{Y}^{(m,1:m-1)}(\mathbf{Y}^{(1:m-1,1:m-1)})^{-1}\mathbf{Y}^{(1:m-1,m)}, \quad m = 2, \dots, M. \quad (237)$$

$$\left[\sigma^2 \mathbf{I} + \sum_{i=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H \right]^{(m,m)} - \Omega^{(m,1:m-1)}(\Omega^{(1:m-1,1:m-1)})^{-1}\Omega^{(1:m-1,m)} \geq (2^{C_m} - 1)\Lambda_m^{(m,m)}, \quad \forall m. \quad (241)$$

$$\left[\sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H \right]^{(\bar{m},\bar{m})} - \Omega^{(\bar{m},1:\bar{m}-1)}(\Omega^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1}\Omega^{(1:\bar{m}-1,\bar{m})} > (2^{C_{\bar{m}}} - 1)\Lambda_{\bar{m}}^{(\bar{m},\bar{m})}. \quad (242)$$

$$\tilde{\Omega}^{(m_1, m_2)} = \begin{cases} \Omega^{(m_1, m_2)}, & \text{if } (m_1, m_2) \neq (\bar{m}, \bar{m}), \\ \Omega^{(\bar{m}, \bar{m})} - \Lambda_{\bar{m}}^{(\bar{m}, \bar{m})} + \tilde{\Lambda}_{\bar{m}}^{(\bar{m}, \bar{m})} > \Omega^{(\bar{m}, \bar{m})}, & \text{otherwise.} \end{cases} \quad (245)$$

Proof: According to (118), we have

$$\Omega^{(m,m)} = \left[\sigma^2 \mathbf{I} + \sum_{i=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H \right]^{(m,m)} + \Lambda_m^{(m,m)}, \quad \forall m. \quad (240)$$

Therefore, we can re-express constraint (120) in problem (119) as (241) on top of the page.

Note that $[\sigma^2 \mathbf{I} + \sum_{i=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H]^{(m,m)}$ and $\Omega^{(m,1:m-1)}(\Omega^{(1:m-1,1:m-1)})^{-1}\Omega^{(1:m-1,m)}$ are not functions of $\Lambda_m^{(m,m)}$, $\forall m$.

Suppose that for a given feasible solution to problem (119), denoted by β_k 's and $\Lambda_m^{(m,m)}$'s, there exists an $1 \leq \bar{m} \leq M$ such that (242) on top of the page is true. Then, we consider another solution where $\tilde{\beta}_k = \beta_k$, $\forall k$, and $\tilde{\Lambda}_m^{(m,m)} = \Lambda_m^{(m,m)}$, $\forall m \neq \bar{m}$, while

$$\begin{aligned} \tilde{\Lambda}_{\bar{m}}^{(\bar{m},\bar{m})} &= \frac{\left[\sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H \right]^{(\bar{m},\bar{m})}}{2^{C_{\bar{m}}} - 1} \\ &\quad - \frac{\tilde{\Omega}^{(\bar{m},1:\bar{m}-1)}(\tilde{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1}\tilde{\Omega}^{(1:\bar{m}-1,\bar{m})}}{2^{C_{\bar{m}}} - 1}, \end{aligned} \quad (243)$$

$$\tilde{\Omega} = \sigma^2 \mathbf{I} + \sum_{k=1}^K \tilde{\beta}_k \mathbf{h}_k \mathbf{h}_k^H + \text{diag}(\tilde{\Lambda}_1^{(1,1)}, \dots, \tilde{\Lambda}_M^{(M,M)}). \quad (244)$$

As a result, $\tilde{\Lambda}_{\bar{m}}^{(\bar{m},\bar{m})} > \Lambda_{\bar{m}}^{(\bar{m},\bar{m})}$.

Consider the above new solution. First, it can be shown that (245) on top of the page is true. As a result, at the new

solution, constraint (120) still holds $\forall m \leq \bar{m}$. For $m > \bar{m}$, we have

$$\begin{aligned} & 2^{C_m} \tilde{\Lambda}_m^{(m,m)} \\ &= 2^{C_m} \Lambda_m^{(m,m)} \\ &\leq \Omega^{(m,m)} - \Omega^{(m,1:m-1)}(\Omega^{(1:m-1,1:m-1)})^{-1}\Omega^{(1:m-1,m)} \\ &= \tilde{\Omega}^{(m,m)} - \tilde{\Omega}^{(m,1:m-1)}(\tilde{\Omega}^{(1:m-1,1:m-1)})^{-1}\tilde{\Omega}^{(1:m-1,m)} \\ &\leq \tilde{\Omega}^{(m,m)} - \tilde{\Omega}^{(m,1:m-1)}(\tilde{\Omega}^{(1:m-1,1:m-1)})^{-1}\tilde{\Omega}^{(1:m-1,m)}, \end{aligned} \quad (246)$$

where (246) is because $\tilde{\Omega} \succeq \Omega$.

Next, consider constraint (107) in problem (119). Since $\tilde{\Lambda}_m^{(m,m)} = \Lambda_m^{(m,m)}$, $\forall m \neq \bar{m}$, $\tilde{\Lambda}_{\bar{m}}^{(\bar{m},\bar{m})} > \Lambda_{\bar{m}}^{(\bar{m},\bar{m})}$, and $\sum_{k=1}^K |\tilde{u}_{k,\bar{m}}|^2 > 0$ according to (52), constraint (107) in problem (119) is satisfied with inequality at least for one k .

To summarize, the new solution is a feasible solution to problem (119), with constraint (107) satisfied with inequality. Thus, given $\tilde{\Lambda}_m^{(m,m)}$'s, we can further find a better solution of β_k 's such that the objective value of problem (119) is enhanced while constraint (107) is satisfied with equality. This indicates that if (239) does not hold for problem (119), we can always find a better solution. Lemma 7 is thus proved. ■

Given any feasible solution $\{\beta_k, \Lambda_m^{(m,m)}\}$ to problem (119) that satisfies the optimal condition (239), we can construct a solution to problem (230) as follows. First, β_k 's and $\Lambda_m^{(m,m)}$'s are unchanged in problem (230). Second, \mathbf{A}_1 is set as

$$\mathbf{A}_1 = \Omega \succ \mathbf{0}. \quad (247)$$

Since β_k 's and $\Lambda_m^{(m,m)}$'s is a feasible solution to problem (119), it must satisfy constraints (107) and (109). Further, it can be observed from (118) that $\Omega \succ \mathbf{0}$. As a result, it follows that

$$\Omega^{(m,m)} - \Omega^{(m,1:m-1)} (\Omega^{(1:m-1,1:m-1)})^{-1} \Omega^{(1:m-1,m)} > 0, \forall m. \quad (248)$$

According to (239), we have $\Lambda_m^{(m,m)} > 0, \forall m$. In other words, constraint (121) holds. Moreover, (247) guarantees that constraint (112) holds in problem (230). At last, (238) and (239) guarantee that

$$\begin{aligned} & \Omega^{(m,m)} - \Omega^{(m,1:m-1)} (\Omega^{(1:m-1,1:m-1)})^{-1} \Omega^{(1:m-1,m)} \\ &= \mathbf{A}_1^{(m,m)} - \mathbf{A}_1^{(m,1:m-1)} (\mathbf{A}_1^{(1:m-1,1:m-1)})^{-1} \mathbf{A}_1^{(1:m-1,m)} \\ &= 2^{C_m} \Lambda_m^{(m,m)}, \forall m. \end{aligned} \quad (249)$$

As a result, constraint (229) also holds. To summarize, given any feasible solution to problem (119) that satisfies (239), we can also generate a feasible solution to problem (230). Moreover, according to Lemma 7, the optimal solution to problem (119) must satisfy condition (239). As a result, the optimal value of problem (119) is achievable for problem (230). In other words, the optimal value of problem (119) is no larger than that of problem (230).

To summarize, we have shown that the optimal value of problem (119) is not smaller than that of problem (230), and at the same time, the optimal value of problem (119) is no larger than that of problem (230). This indicates that problem (119) is equivalent to problem (230), which is further equivalent to problem (111). In other words, problem (119) is equivalent to (111). This completes the proof of Proposition 8.

H. Proof of Proposition 9

According to Lemma 7 in Appendix G, with the optimal solution to problem (119), all the constraints shown in (120) should hold with equality. In the following, we show that with the optimal solution, to problem (119), all the constraints shown in (107) should hold with equality, i.e.,

$$\begin{aligned} & \sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 \\ & - \frac{\beta_k |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_k} - 1} = 0, \forall k. \end{aligned} \quad (250)$$

Given any feasible solution to problem (119), suppose that there exists a \bar{k} such that

$$\begin{aligned} & \sigma^2 + \sum_{j \neq \bar{k}} \beta_j |\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{\bar{k},m}|^2 \\ & - \frac{\beta_{\bar{k}} |\bar{\mathbf{u}}_{\bar{k}}^H \mathbf{h}_{\bar{k}}|^2}{2^{R_{\bar{k}}} - 1} > 0. \end{aligned} \quad (251)$$

Then, consider another solution of $\tilde{\beta}_{\bar{k}}$ such that

$$\begin{aligned} & \sigma^2 + \sum_{j \neq \bar{k}} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \Lambda_m^{(m,m)} |\bar{u}_{k,m}|^2 \\ & - \frac{\tilde{\beta}_{\bar{k}} |\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}{2^{R_{\bar{k}}} - 1} = 0. \end{aligned} \quad (252)$$

As a result, we have

$$\tilde{\beta}_{\bar{k}} > \beta_{\bar{k}}. \quad (253)$$

First, it can be shown that with $\tilde{\beta}_{\bar{k}}$, constraint (107) holds for all k . Moreover, according to (118), it follows that

$$\begin{aligned} \Omega &= \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \sum_{m=1}^M \mathbf{E}_m \Lambda_m \mathbf{E}_m \\ &\preceq \sigma^2 \mathbf{I} + \sum_{j \neq \bar{k}} \beta_k \mathbf{h}_k \mathbf{h}_k^H + \tilde{\beta}_{\bar{k}} \mathbf{h}_{\bar{k}} \mathbf{h}_{\bar{k}}^H + \sum_{m=1}^M \mathbf{E}_m \Lambda_m \mathbf{E}_m. \end{aligned} \quad (254)$$

According to Lemma 6, it follows that the new solution also satisfies constraint (120). As a result, the new solution is also a feasible solution to problem (119). However, with the new solution, the objective value of problem (119) is increased. This indicates that with the optimal solution to problem (119), all the constraints shown in (107) should hold with equality.

Proposition 9 is thus proved.

I. Proof of Proposition 10

Given any $\beta = [\beta_1, \dots, \beta_K] > \mathbf{0}$, let $\lambda(\beta) = [\lambda_1(\beta), \dots, \lambda_M(\beta)]$ denote the solution of $\Lambda_m^{(m,m)}$'s (i.e., $\Lambda_m^{(m,m)} = \lambda_m(\beta), \forall m$) that satisfies constraint (123) in problem (124). The uniqueness of $\lambda(\beta)$ given $\beta > \mathbf{0}$ can be shown as follows. First, it follows from (123) that

$$\lambda_1(\beta) = \frac{\sigma^2 + \sum_{k=1}^K \beta_k h_{k,1} h_{k,1}^H}{2^{C_1} - 1}. \quad (255)$$

Next, it can be shown that if $\lambda_1(\beta), \dots, \lambda_{m-1}(\beta)$, then $\lambda_m(\beta)$ can be uniquely determined as

$$\begin{aligned} \lambda_m(\beta) &= \frac{\sigma^2 + \sum_{k=1}^K \beta_k h_{k,m} h_{k,m}^H}{2^{C_m} - 1} \\ & - \frac{\Omega^{(m,1:m-1)} (\Omega^{(1:m-1,1:m-1)})^{-1} \Omega^{(1:m-1,m)}}{2^{C_m} - 1}, \end{aligned} \quad (256)$$

since $\Omega^{(m,1:m-1)}$, $\Omega^{(1:m-1,1:m-1)}$, and $\Omega^{(1:m-1,m)}$ only depend on $\lambda_1(\beta), \dots, \lambda_{m-1}(\beta)$. As a result, given any $\beta > \mathbf{0}$, there always exists a unique solution $\lambda(\beta)$ such that constraint (123) is satisfied in problem (124). Moreover, the following lemma shows some important properties of $\lambda(\beta)$.

Lemma 8: Given $\beta \geq \mathbf{0}$, the function $\lambda(\beta)$ defined by (255) and (256) has the following properties:

1. $\lambda(\beta) > \mathbf{0}$;
2. Given any $\alpha > 1$, it follows that $\lambda(\alpha\beta) < \alpha\lambda(\beta)$;
3. If $\tilde{\beta} \geq \beta$, then $\lambda(\tilde{\beta}) \geq \lambda(\beta)$.

$$\lambda_{\bar{m}}(\alpha\beta) = \frac{\sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,\bar{m}} h_{k,\bar{m}}^H - \bar{\Omega}^{(\bar{m},1:\bar{m}-1)} (\bar{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1} \bar{\Omega}^{(1:\bar{m}-1,\bar{m})}}{2^{C_{\bar{m}}} - 1} \quad (267)$$

$$= \frac{\sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,\bar{m}} h_{k,\bar{m}}^H - \alpha^2 \hat{\Omega}^{(\bar{m},1:\bar{m}-1)} (\hat{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1} \hat{\Omega}^{(1:\bar{m}-1,\bar{m})}}{2^{C_{\bar{m}}} - 1} \quad (268)$$

$$< \frac{\sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,\bar{m}} h_{k,\bar{m}}^H - \alpha^2 \hat{\Omega}^{(\bar{m},1:\bar{m}-1)} (\alpha \hat{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1} \hat{\Omega}^{(1:\bar{m}-1,\bar{m})}}{2^{C_{\bar{m}}} - 1} \quad (269)$$

$$< \frac{\alpha \left(\sigma^2 + \sum_{k=1}^K \beta_k h_{k,\bar{m}} h_{k,\bar{m}}^H - \hat{\Omega}^{(\bar{m},1:\bar{m}-1)} (\hat{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)})^{-1} \hat{\Omega}^{(1:\bar{m}-1,\bar{m})} \right)}{2^{C_{\bar{m}}} - 1} \quad (270)$$

$$= \alpha \lambda_{\bar{m}}(\beta). \quad (271)$$

Proof: First, since $\lambda_m(\beta)$'s in (255) and (256) satisfy (123) and $\Omega \succ \mathbf{0}$, it then follows that $\lambda_m(\beta) > 0, \forall m$.

Next, we show that given any $\alpha > 1$, we have $\lambda(\alpha\beta) < \alpha\lambda(\beta)$. For convenience, we define

$$\hat{\Omega} = \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k \mathbf{h}_k^H + \text{diag}(\lambda_1(\beta), \dots, \lambda_M(\beta)), \quad (257)$$

$$\bar{\Omega} = \sigma^2 \mathbf{I} + \sum_{k=1}^K \alpha \beta_k \mathbf{h}_k \mathbf{h}_k^H + \text{diag}(\lambda_1(\alpha\beta), \dots, \lambda_M(\alpha\beta)), \quad (258)$$

as the corresponding Ω 's shown in (118) with β and $\Lambda_m^{(m,m)}$'s replaced by β and $\lambda(\beta)$ as well as $\alpha\beta$ and $\lambda(\alpha\beta)$, respectively. It is observed from (257) and (258) that the non-diagonal elements of $\hat{\Omega}$ and $\bar{\Omega}$ satisfy

$$\alpha \hat{\Omega}^{(m1,m2)} = \bar{\Omega}^{(m1,m2)}, \quad \forall m1 \neq m2. \quad (259)$$

As a result, we have

$$\bar{\Omega}^{(m,1:m-1)} = \alpha \hat{\Omega}^{(m,1:m-1)}, \quad \forall m. \quad (260)$$

In the following, we shown by induction that $\alpha\lambda_m(\beta) > \lambda_m(\alpha\beta), \forall m$.

First, when $m = 1$, it follows from (255) that

$$\begin{aligned} \alpha\lambda_1(\beta) &= \frac{\alpha\sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,1} h_{k,1}^H}{2^{C_1} - 1} \\ &> \frac{\sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,1} h_{k,1}^H}{2^{C_1} - 1} \\ &= \lambda_1(\alpha\beta). \end{aligned} \quad (261)$$

Next, we shown that given any $\bar{m} \geq 2$, if

$$\alpha\lambda_m(\beta) > \lambda_m(\alpha\beta), \quad \forall m = 1, \dots, \bar{m} - 1, \quad (262)$$

then

$$\alpha\lambda_{\bar{m}}(\beta) > \lambda_{\bar{m}}(\alpha\beta). \quad (263)$$

Given (262), the diagonal elements of $\hat{\Omega}$ and $\bar{\Omega}$ defined in (257) and (258) satisfy

$$\begin{aligned} \bar{\Omega}^{(m,m)} &= \sigma^2 + \alpha \sum_{k=1}^K \beta_k h_{k,m} h_{k,m}^H + \lambda_m(\alpha\beta) \\ &< \alpha \left(\sigma^2 + \sum_{k=1}^K \beta_k h_{k,m} h_{k,m}^H + \lambda_m(\beta) \right) \\ &= \alpha \hat{\Omega}^{(m,m)}, \quad \forall m = 1, \dots, \bar{m} - 1. \end{aligned} \quad (264)$$

Based on (260) and (264), it then follows that

$$\bar{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)} \prec \alpha \hat{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)}, \quad (265)$$

or

$$\left(\bar{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)} \right)^{-1} \succ \left(\alpha \hat{\Omega}^{(1:\bar{m}-1,1:\bar{m}-1)} \right)^{-1}. \quad (266)$$

Last, it follows that (267) – (271) on top of the page are true, where (268) is due to (260) and (269) is due to (266). As a result, given any $\bar{m} \geq 2$, if (262) is true, then (263) is true. By combining the above with (261), it follows that $\alpha\lambda_m(\beta) > \lambda_m(\alpha\beta), \forall m$. Next, we shown that if $\bar{\beta} = [\bar{\beta}_1, \dots, \bar{\beta}_K]^T$ satisfies $\bar{\beta}_k \geq \beta_k, \forall k$, then $\lambda(\bar{\beta}) \geq \lambda(\beta)$ by induction. First, it can be shown from (255) that if $\bar{\beta} \geq \beta$, then

$$\lambda_1(\bar{\beta}) \geq \lambda_1(\beta). \quad (272)$$

In the following, we prove that given any $\bar{m} \geq 2$, if $\lambda_m(\bar{\beta}) \geq \lambda_m(\beta), \forall m \leq \bar{m} - 1$, then $\lambda_{\bar{m}}(\bar{\beta}) \geq \lambda_{\bar{m}}(\beta)$. To prove this, given any $\bar{m} \geq 2$, define

$$\begin{aligned} \mathbf{X}_{\bar{m}} &= \sigma^2 \mathbf{I} + \sum_{k=1}^K \beta_k \mathbf{h}_k^{(1:\bar{m})} [\mathbf{h}_k^{(1:\bar{m})}]^H \\ &\quad + \text{diag}([\lambda_1(\beta), \dots, \lambda_{\bar{m}-1}(\beta), 0]) \in \mathbb{C}^{\bar{m} \times \bar{m}}, \end{aligned} \quad (273)$$

$$\begin{aligned} \bar{\mathbf{X}}_{\bar{m}} &= \sigma^2 \mathbf{I} + \sum_{k=1}^K \bar{\beta}_k \mathbf{h}_k^{(1:\bar{m})} [\mathbf{h}_k^{(1:\bar{m})}]^H \\ &\quad + \text{diag}([\lambda_1(\bar{\beta}), \dots, \lambda_{\bar{m}-1}(\bar{\beta}), 0]) \in \mathbb{C}^{\bar{m} \times \bar{m}}. \end{aligned} \quad (274)$$

$$I_k(\boldsymbol{\beta}) = \frac{(2^{R_k} - 1)(\sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m(\boldsymbol{\beta}) |\bar{u}_{k,m}|^2)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2}, \quad \forall k. \quad (277)$$

With $\bar{\boldsymbol{\beta}} \geq \boldsymbol{\beta}$ and $\lambda_m(\bar{\boldsymbol{\beta}}) \geq \lambda_m(\boldsymbol{\beta})$, $\forall m \leq \bar{m} - 1$, it follows that $\bar{\mathbf{X}}_{\bar{m}} \succeq \mathbf{X}_{\bar{m}}$. Based on Lemma 6, it can be shown from (256) that

$$\begin{aligned} & \lambda_{\bar{m}}(\bar{\boldsymbol{\beta}}) \\ &= \frac{\bar{\mathbf{X}}_{\bar{m}}^{(\bar{m}, \bar{m})} - \bar{\mathbf{X}}_{\bar{m}}^{(1:\bar{m}-1, \bar{m})} (\bar{\mathbf{X}}_{\bar{m}}^{(1:\bar{m}-1, 1:\bar{m}-1)})^{-1} \bar{\mathbf{X}}_{\bar{m}}^{(\bar{m}, 1:\bar{m}-1)}}{2^{\bar{m}} - 1} \\ &\geq \frac{\mathbf{X}_{\bar{m}}^{(\bar{m}, \bar{m})} - \mathbf{X}_{\bar{m}}^{(1:\bar{m}-1, \bar{m})} (\mathbf{X}_{\bar{m}}^{(1:\bar{m}-1, 1:\bar{m}-1)})^{-1} \mathbf{X}_{\bar{m}}^{(\bar{m}, 1:\bar{m}-1)}}{2^{\bar{m}} - 1} \\ &= \lambda_{\bar{m}}(\boldsymbol{\beta}). \end{aligned} \quad (275)$$

To summarize, we have shown that $\lambda_1(\bar{\boldsymbol{\beta}}) \geq \lambda_1(\boldsymbol{\beta})$ and given any $\bar{m} \geq 2$, $\lambda_{\bar{m}}(\bar{\boldsymbol{\beta}}) \geq \lambda_{\bar{m}}(\boldsymbol{\beta})$ if $\lambda_m(\bar{\boldsymbol{\beta}}) \geq \lambda_m(\boldsymbol{\beta})$, $\forall m \leq \bar{m} - 1$. Therefore, by deduction, it can be shown that $\boldsymbol{\lambda}(\bar{\boldsymbol{\beta}}) \geq \boldsymbol{\lambda}(\boldsymbol{\beta})$ if $\bar{\boldsymbol{\beta}} \geq \boldsymbol{\beta}$.

Lemma 8 is thus proved. \blacksquare

Then, the remaining job is to show that there is only a unique solution $\boldsymbol{\beta} > \mathbf{0}$ such that $\boldsymbol{\beta} > \mathbf{0}$ and $\boldsymbol{\lambda}(\boldsymbol{\beta})$ satisfy constraint (122) in problem (124). In the following, we prove this.

Note that constraint (122) can be re-expressed as

$$\boldsymbol{\beta} = \mathbf{I}(\boldsymbol{\beta}), \quad (276)$$

where $\mathbf{I}(\boldsymbol{\beta}) = [I_1(\boldsymbol{\beta}), \dots, I_K(\boldsymbol{\beta})]$ with $I_k(\boldsymbol{\beta})$ defined in (277) on top of the page.

In the following, we show three important properties of the function $\mathbf{I}(\boldsymbol{\beta})$.

Lemma 9: Given $\boldsymbol{\beta} \geq \mathbf{0}$, the function $\mathbf{I}(\boldsymbol{\beta})$ defined by (277) satisfies the following three properties:

1. $\mathbf{I}(\boldsymbol{\beta}) > \mathbf{0}$;
2. Given any $\alpha > 1$, it follows that $\mathbf{I}(\alpha\boldsymbol{\beta}) < \alpha\mathbf{I}(\boldsymbol{\beta})$;
3. If $\bar{\boldsymbol{\beta}} \geq \boldsymbol{\beta}$, then $\mathbf{I}(\bar{\boldsymbol{\beta}}) \geq \mathbf{I}(\boldsymbol{\beta})$.

Proof: First, given any $\boldsymbol{\beta} \geq \mathbf{0}$, it can be shown from (277) that

$$\mathbf{I}(\boldsymbol{\beta}) > \mathbf{0}. \quad (278)$$

Next, according to Lemma 8, given any $\alpha > 1$, it can be shown that

$$\begin{aligned} & I_k(\alpha\boldsymbol{\beta}) \\ &= \frac{(2^{R_k} - 1) \left(\sigma^2 + \sum_{j \neq k} \alpha \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m(\alpha\boldsymbol{\beta}) |\bar{u}_{k,m}|^2 \right)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \\ &\leq \frac{\alpha(2^{R_k} - 1) \left(\sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m(\boldsymbol{\beta}) |\bar{u}_{k,m}|^2 \right)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \\ &= \alpha I_k(\boldsymbol{\beta}), \quad \forall k, \end{aligned} \quad (279)$$

where the inequality is due to Lemma 8.

Last, if $\bar{\boldsymbol{\beta}} \geq \boldsymbol{\beta}$, it then follows that

$$\begin{aligned} & I_k(\bar{\boldsymbol{\beta}}) \\ &= \frac{(2^{R_k} - 1) \left(\sigma^2 + \sum_{j \neq k} \bar{\beta}_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m(\bar{\boldsymbol{\beta}}) |\bar{u}_{k,m}|^2 \right)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \\ &\geq \frac{(2^{R_k} - 1) \left(\sigma^2 + \sum_{j \neq k} \beta_j |\bar{\mathbf{u}}_k^H \mathbf{h}_j|^2 + \sum_{m=1}^M \lambda_m(\boldsymbol{\beta}) |\bar{u}_{k,m}|^2 \right)}{|\bar{\mathbf{u}}_k^H \mathbf{h}_k|^2} \\ &= I_k(\boldsymbol{\beta}), \quad \forall k, \end{aligned} \quad (280)$$

where the inequality is due to Lemma 8.

Lemma 9 is thus proved. \blacksquare

Lemma 9 shows that the function $\mathbf{I}(\boldsymbol{\beta})$ defined by (277) is a standard interference function [49]. It then follows from [49, Theorem 1] that there exists a unique solution $\boldsymbol{\beta} > \mathbf{0}$ to equation (276). Note that we have shown in the above that given $\boldsymbol{\beta} > \mathbf{0}$, there exists a unique solution $\boldsymbol{\lambda}(\boldsymbol{\beta})$ that satisfies constraint (123). Therefore, if problem (124) is feasible, there exists only one solution to problem (124).

Proposition 10 is thus proved.

J. Proof of Lemma 1

First, it can be easily shown that if $\mathbf{p}^{\text{ul}} \geq \mathbf{0}$, then $I_k(\mathbf{p}^{\text{ul}}) > 0$, $\forall k$. Next, given $\alpha > 1$, it follows from (138) that $q_m^{\text{ul}}(\alpha\mathbf{p}^{\text{ul}}) < \alpha q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$, $\forall m$. As a result, $\forall k \in \mathcal{K}$, we have

$$\begin{aligned} & \sum_{j \neq k} \alpha p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\alpha\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\alpha\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I} \\ &< \alpha \left(\sum_{j \neq k} p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I} \right). \end{aligned} \quad (281)$$

Based on (143), it follows that $I_k(\alpha\mathbf{p}^{\text{ul}}) < \alpha I_k(\mathbf{p}^{\text{ul}})$, $\forall k$, given $\alpha > 1$.

Last, if $\bar{\mathbf{p}}^{\text{ul}} \geq \mathbf{p}^{\text{ul}}$, then based on (138), it follows that $q_m^{\text{ul}}(\bar{\mathbf{p}}^{\text{ul}}) \geq q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$, $\forall m$. As a result, $\forall k \in \mathcal{K}$, we have

$$\begin{aligned} & \sum_{j \neq k} \bar{p}_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\bar{\mathbf{p}}^{\text{ul}}), \dots, q_M^{\text{ul}}(\bar{\mathbf{p}}^{\text{ul}})) + \sigma^2 \mathbf{I} \\ &\geq \sum_{j \neq k} p_j^{\text{ul}} \mathbf{h}_j \mathbf{h}_j^H + \text{diag}(q_1^{\text{ul}}(\mathbf{p}^{\text{ul}}), \dots, q_M^{\text{ul}}(\mathbf{p}^{\text{ul}})) + \sigma^2 \mathbf{I}. \end{aligned} \quad (282)$$

Based on (143), it follows that $I_k(\bar{\mathbf{p}}^{\text{ul}}) > I_k(\mathbf{p}^{\text{ul}})$, $\forall k$.

Lemma 1 is thus proved.

K. Proof of Lemma 2

First, it can be easily shown that if $\mathbf{p}^{\text{ul}} \geq \mathbf{0}$, then $I_k(\mathbf{p}^{\text{ul}}) > 0$, $\forall k \in \mathcal{K}$. Next, given $\alpha > 1$, it follows from Lemma 8 that $\{q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})\}$ defined in (154) and (155) satisfies $q_m^{\text{ul}}(\alpha\mathbf{p}^{\text{ul}}) < \alpha q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$, $\forall m \in \mathcal{M}$. Then, similar to Appendix J, it can be shown that $\{I_k(\mathbf{p}^{\text{ul}})\}$ defined in (158) satisfies $I_k(\alpha\mathbf{p}^{\text{ul}}) < \alpha I_k(\mathbf{p}^{\text{ul}})$, $\forall k \in \mathcal{K}$, given $\alpha > 1$. Last, according to Lemma 8, if $\bar{\mathbf{p}}^{\text{ul}} \geq \mathbf{p}^{\text{ul}}$, then $q_m^{\text{ul}}(\bar{\mathbf{p}}^{\text{ul}}) \geq q_m^{\text{ul}}(\mathbf{p}^{\text{ul}})$, $\forall m \in \mathcal{M}$. Similar to Appendix J, it can be shown that $I_k(\bar{\mathbf{p}}^{\text{ul}}) \geq I_k(\mathbf{p}^{\text{ul}})$, $\forall k \in \mathcal{K}$.

Lemma 2 is thus proved.

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