Asymptotically Free Sketching and Applications in Ridge Regression

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Thanks to collaborators

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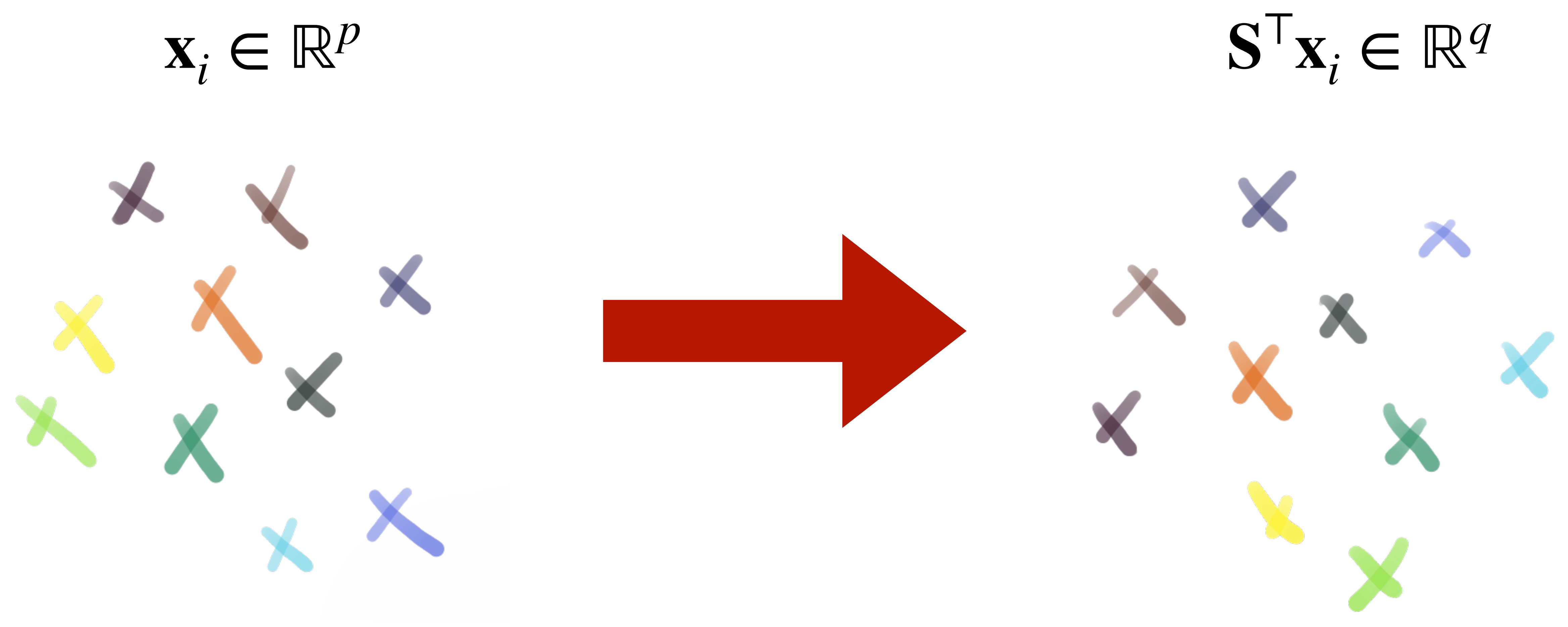

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  - Use sum of one-hot encodings as features: $p \sim 10^7$, feasible

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  - $\mathbf{S} = \mathbf{X}^\top$ or $\mathbf{S} = \mathbf{Z}^\top$ is determined by nature and may not be observed

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Q: How does sketching affect the result in machine learning?

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- Typical usage: $\mathbf{A}_n$ is complicated and $\mathbf{B}_n$ is simple, but $\mathbf{A}_n \simeq \mathbf{B}_n$, so we can use $\mathbf{B}_n$ to understand $\mathbf{A}_n$

Asymptotic equivalences

- Also known as "deterministic equivalences" in random matrix theory

- **Definition.** Two sequences of matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ are *asymptotically equivalent,* written $\mathbf{A}_n \simeq \mathbf{B}_n$, if for any sequence $\mathbf{\Theta}_n$ independent of $\mathbf{A}_n$ and $\mathbf{B}_n$ with bounded trace norm,

$$\lim_{n\to\infty} \mathrm{tr}[\mathbf{\Theta}_n(\mathbf{A}_n - \mathbf{B}_n)] = 0 \quad \text{almost surely.}$$

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- $\mathbf{A}_n \simeq \mathbf{B}_n$ is analogous to $\mathbb{E}[\mathbf{A}] = \mathbb{E}[\mathbf{B}]$, but single-instance

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    - Called "pseudoinverse" because when $\mathbf{S}$ has orthonormal columns and $\lambda \to 0$, it is the Moore–Penrose pseudoinverse of $\mathbf{S}\mathbf{S}^\top \mathbf{A}\mathbf{S}\mathbf{S}^\top$

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- **Less distortion:** $\lambda < \gamma < \mu$ for i.i.d. sketch when $\mu > 0$

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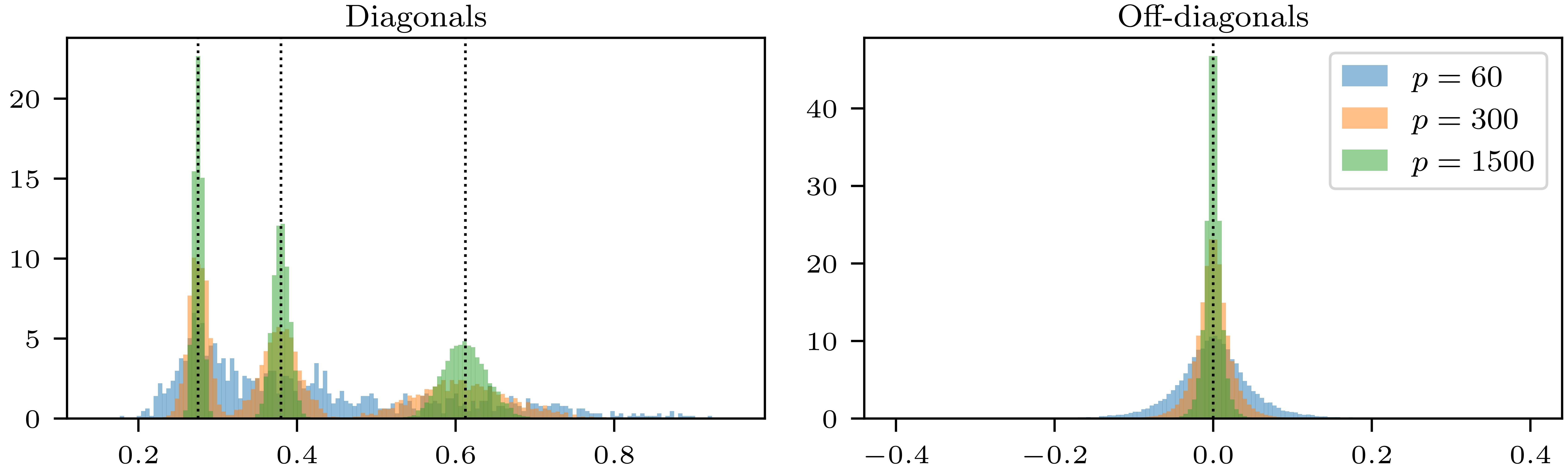

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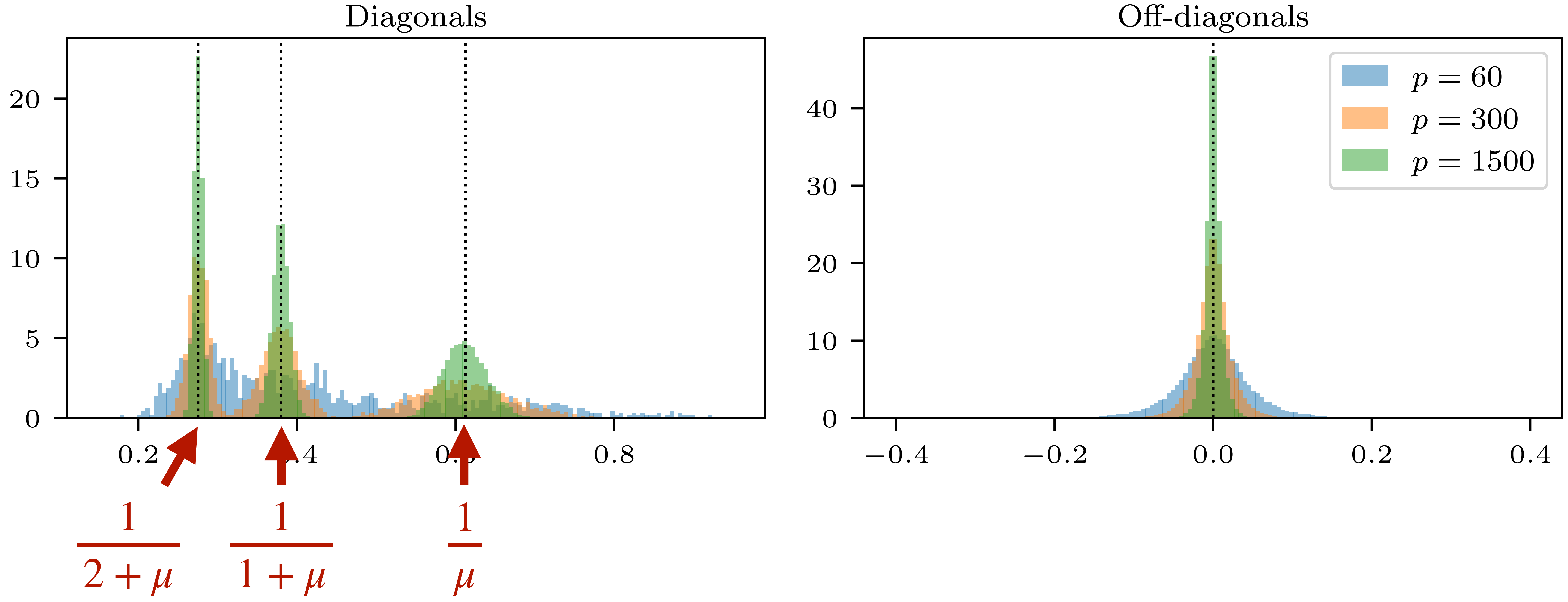

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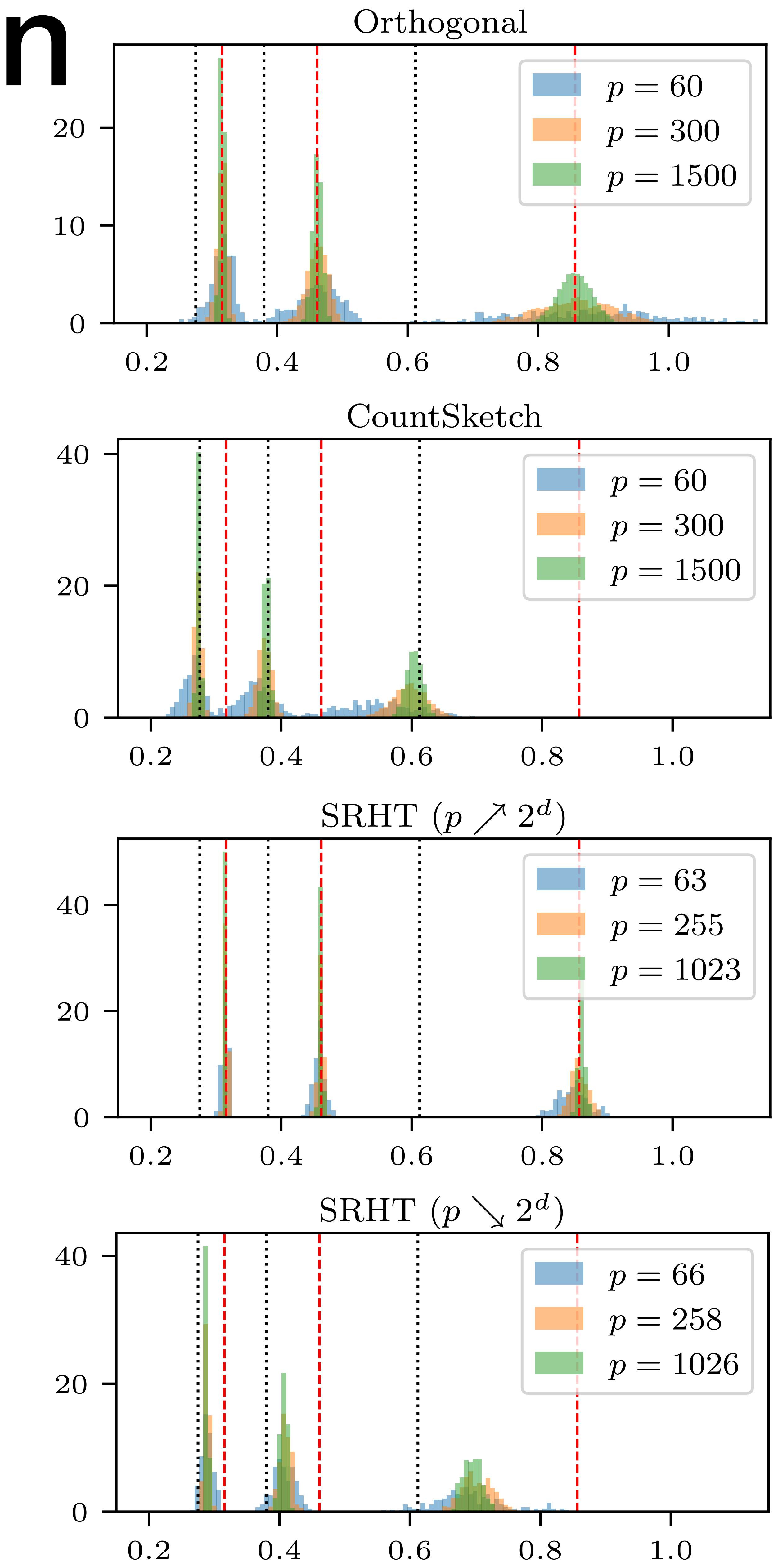

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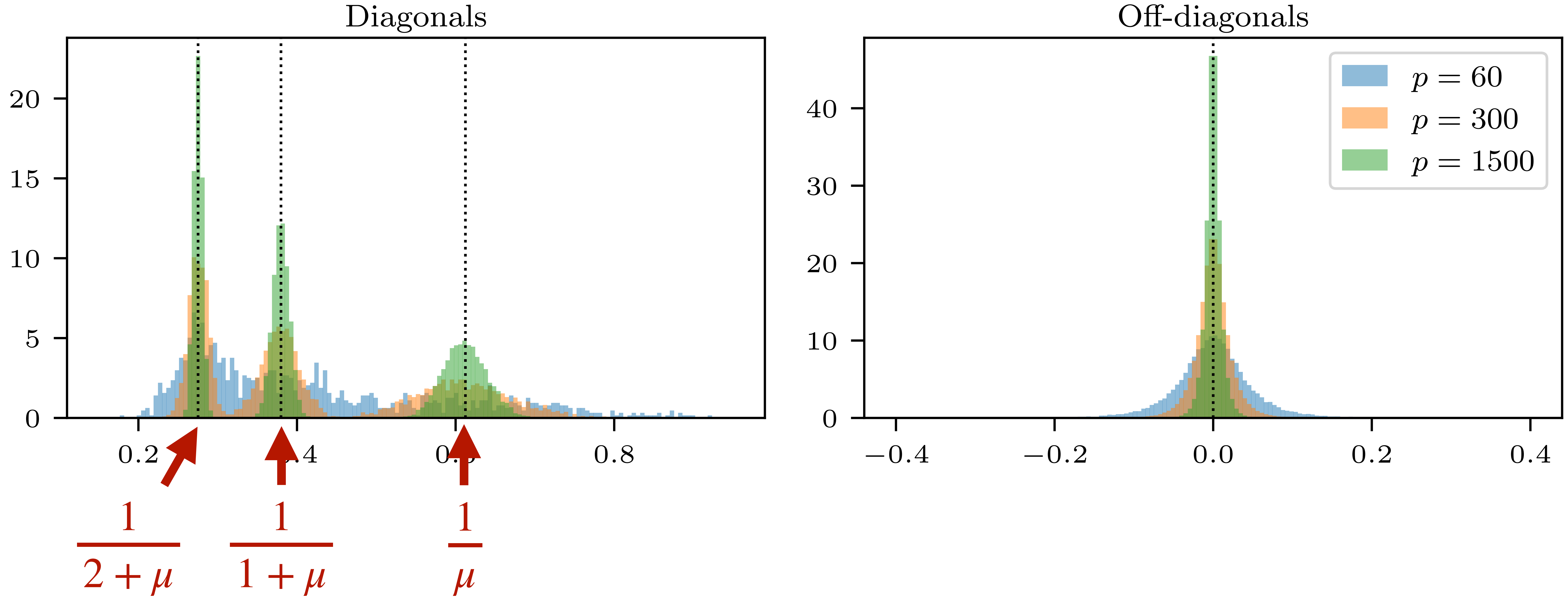

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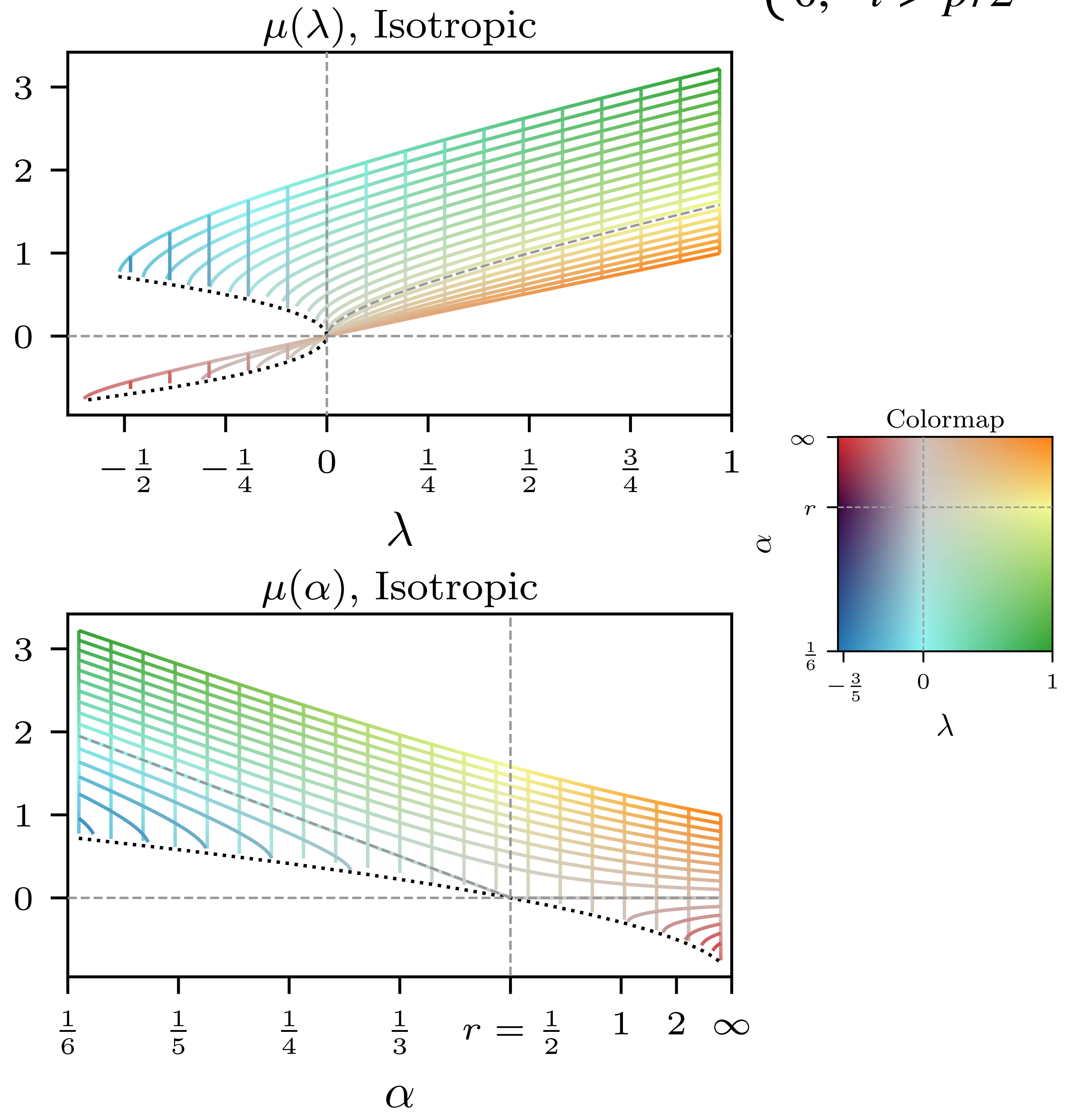

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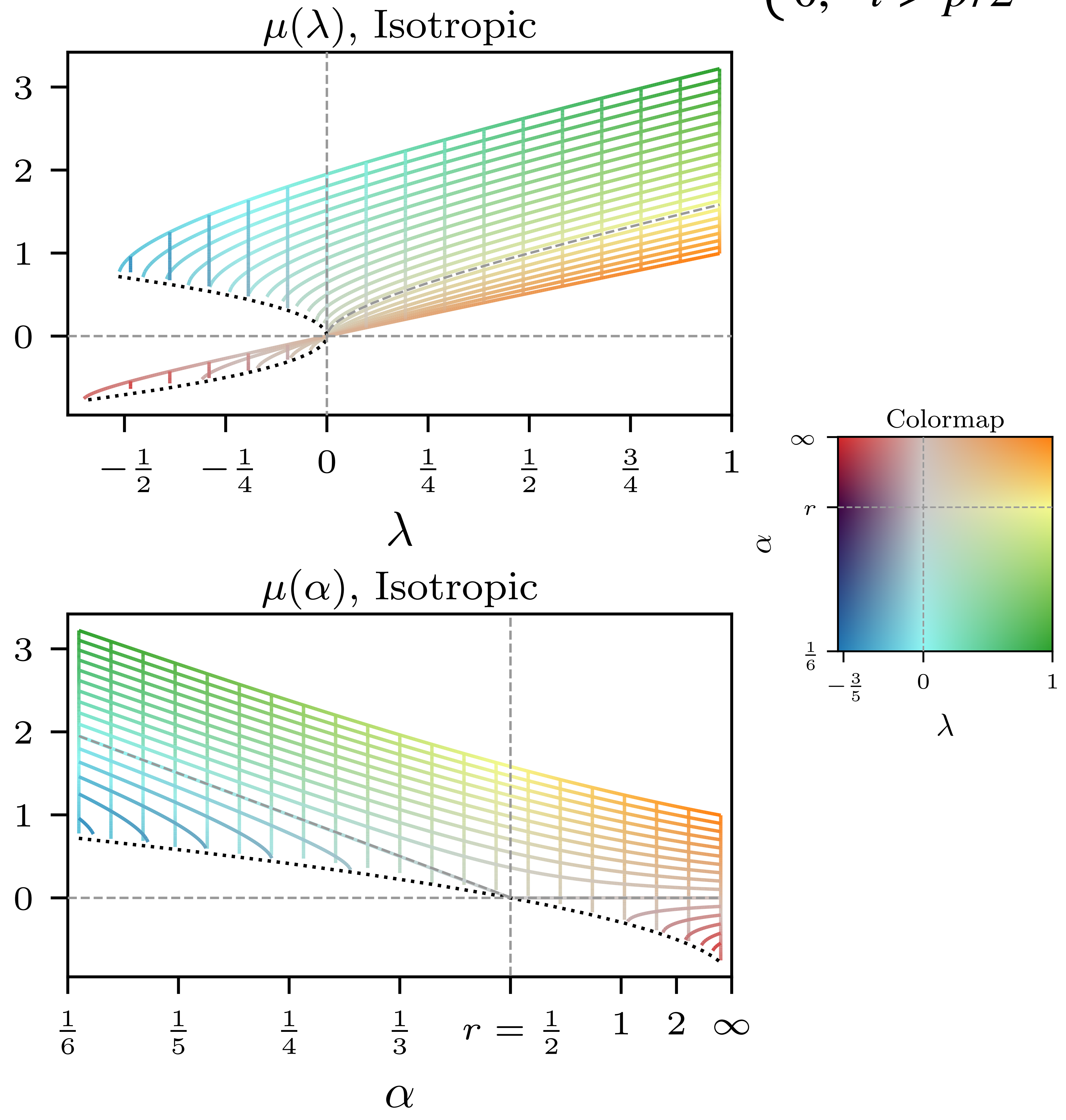

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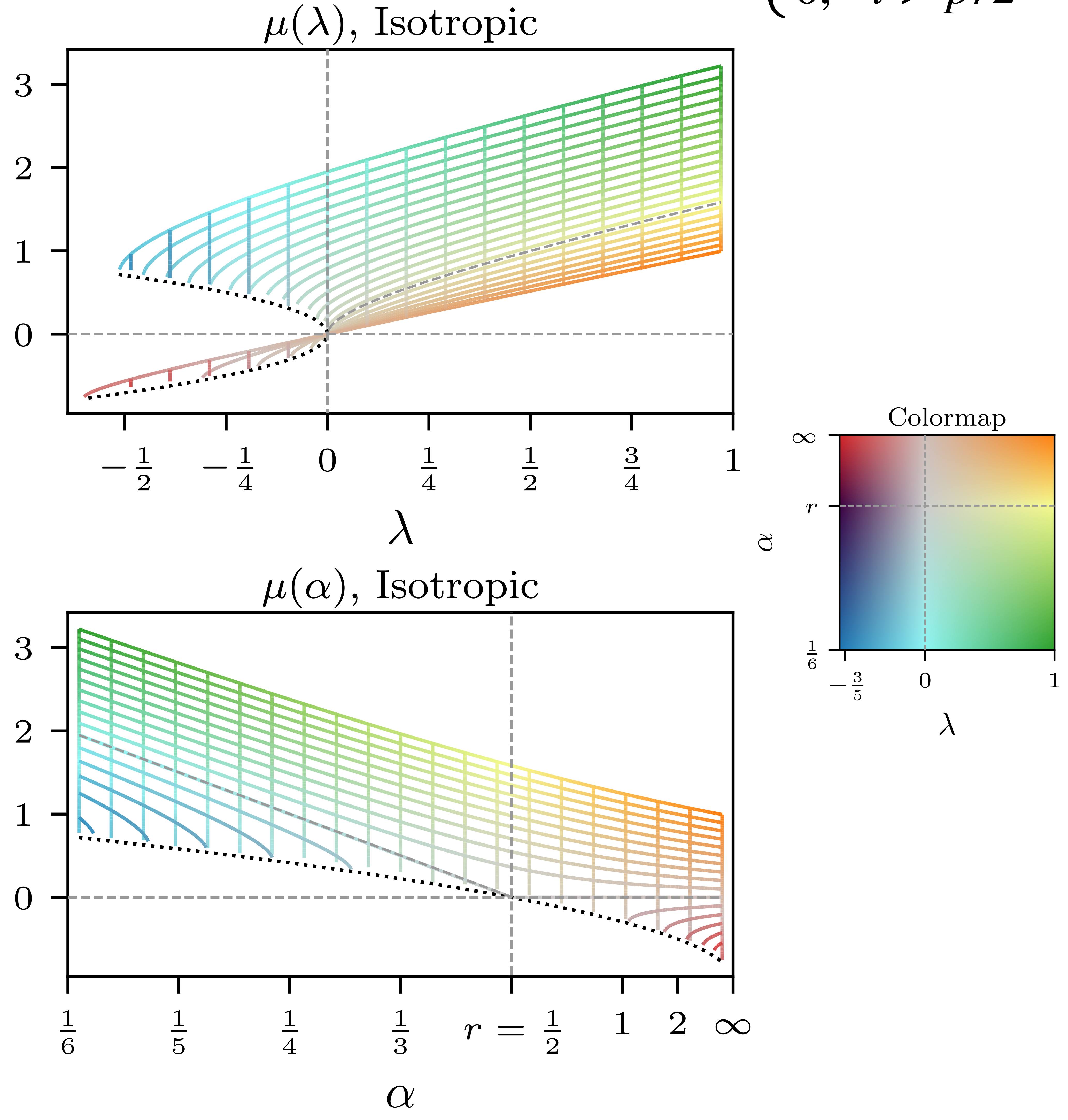

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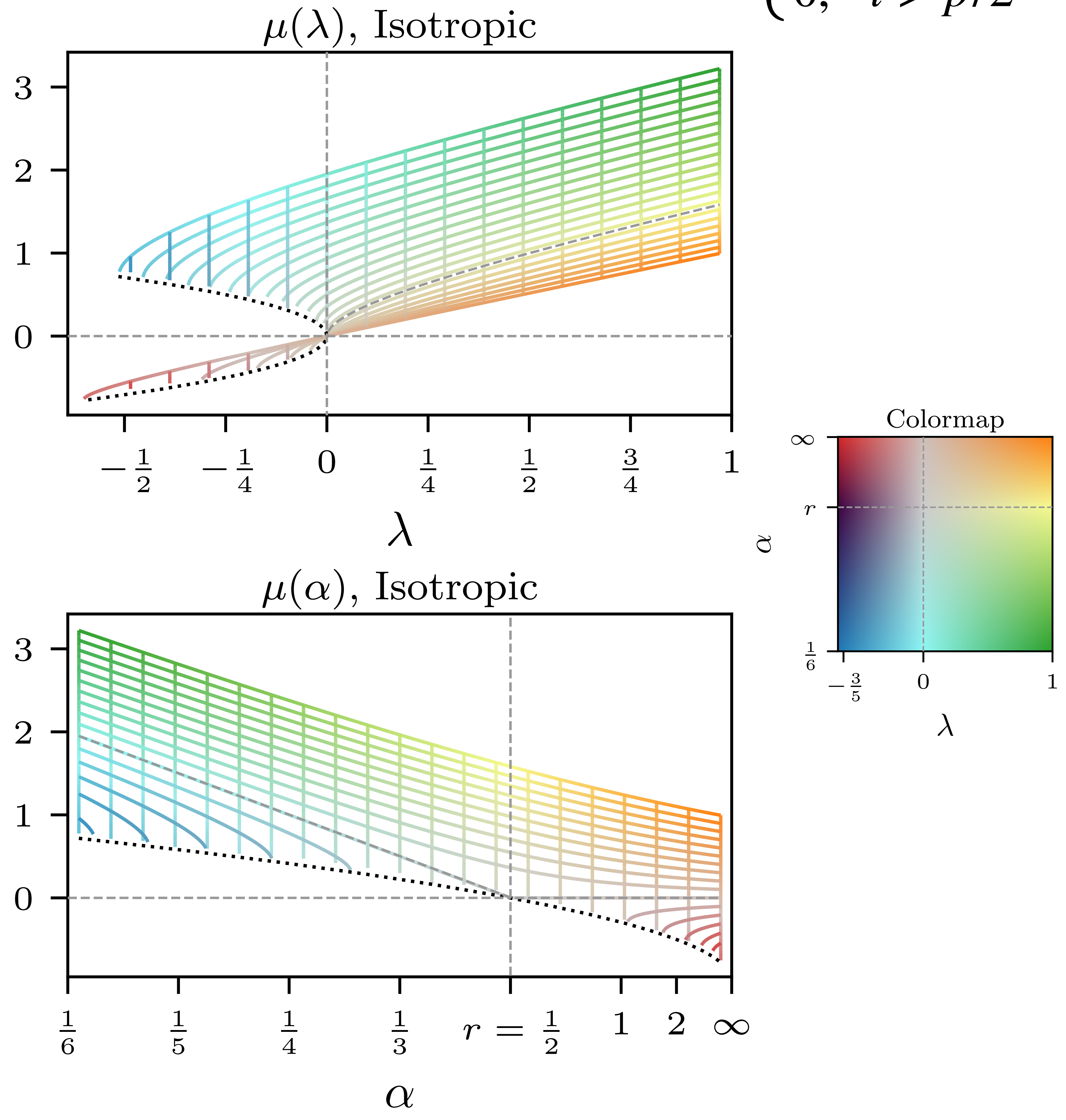

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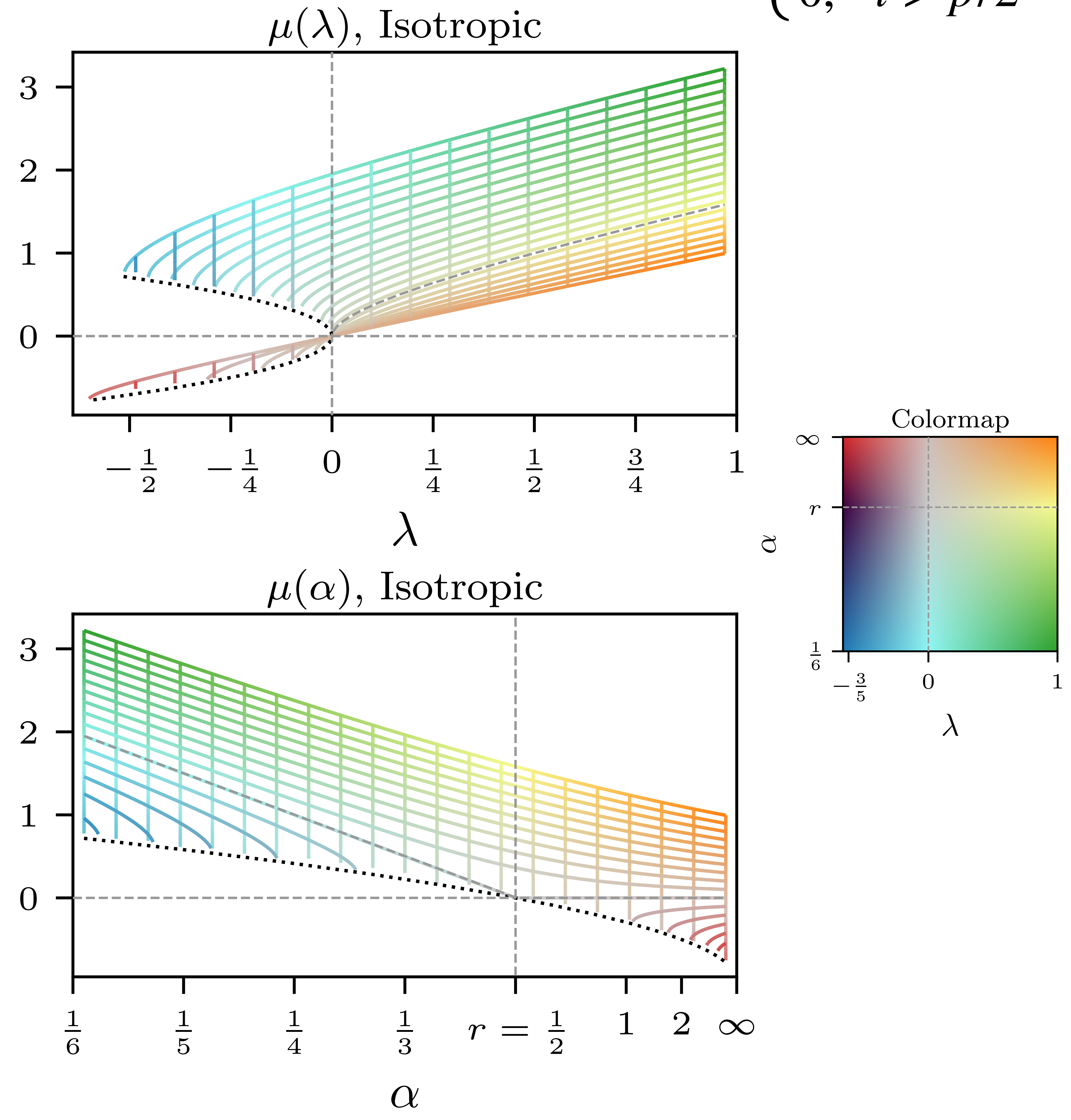

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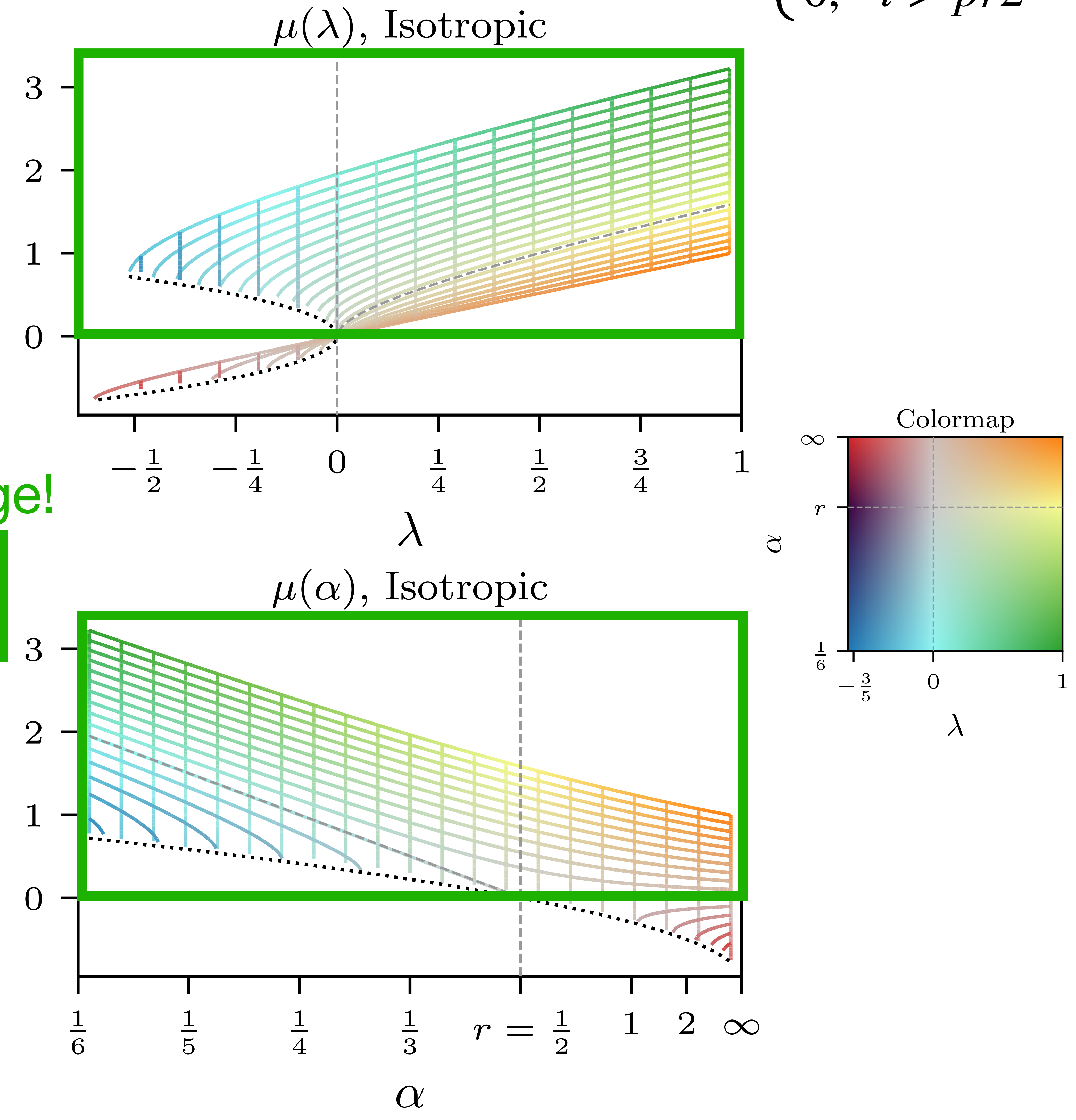

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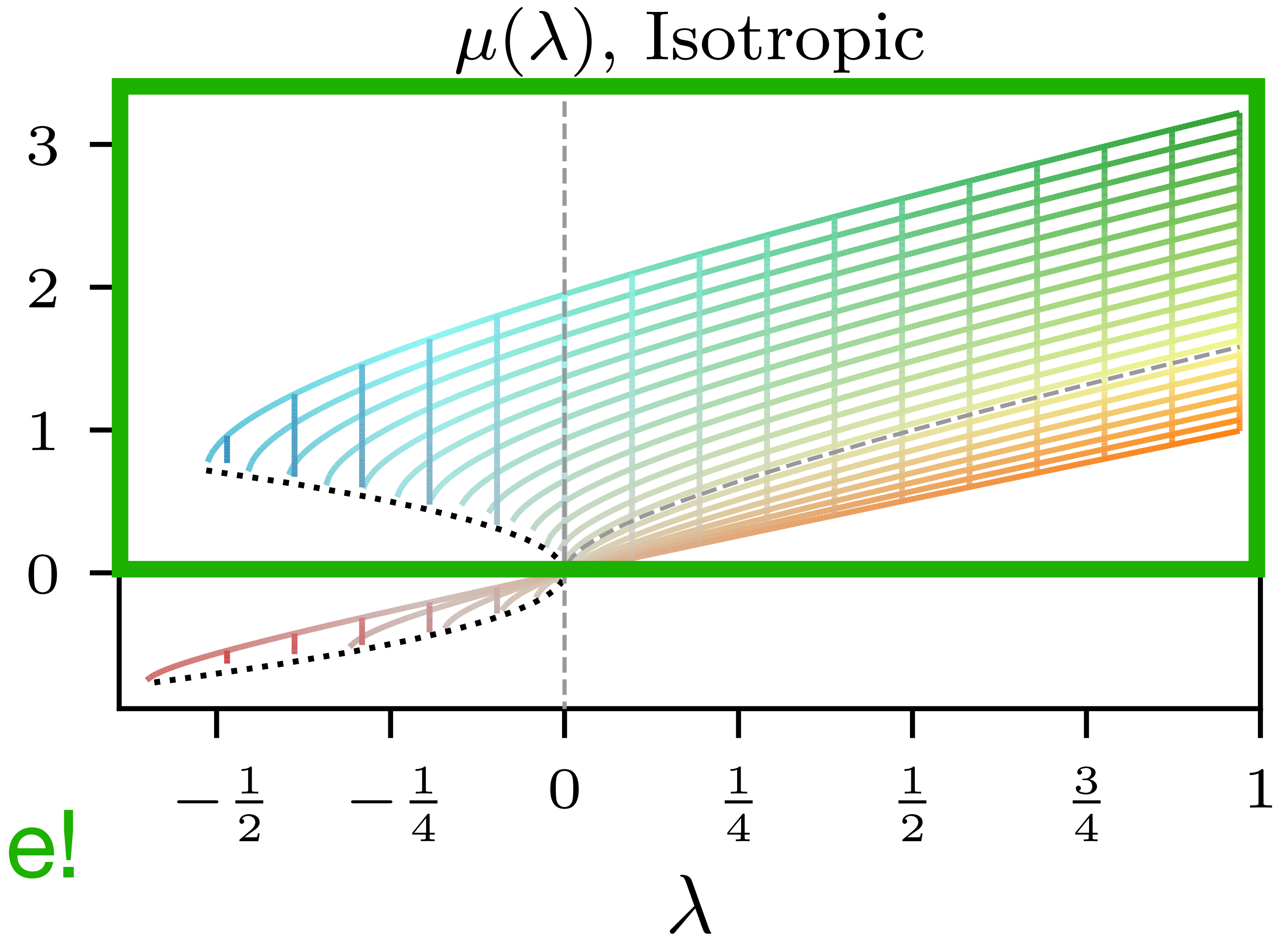

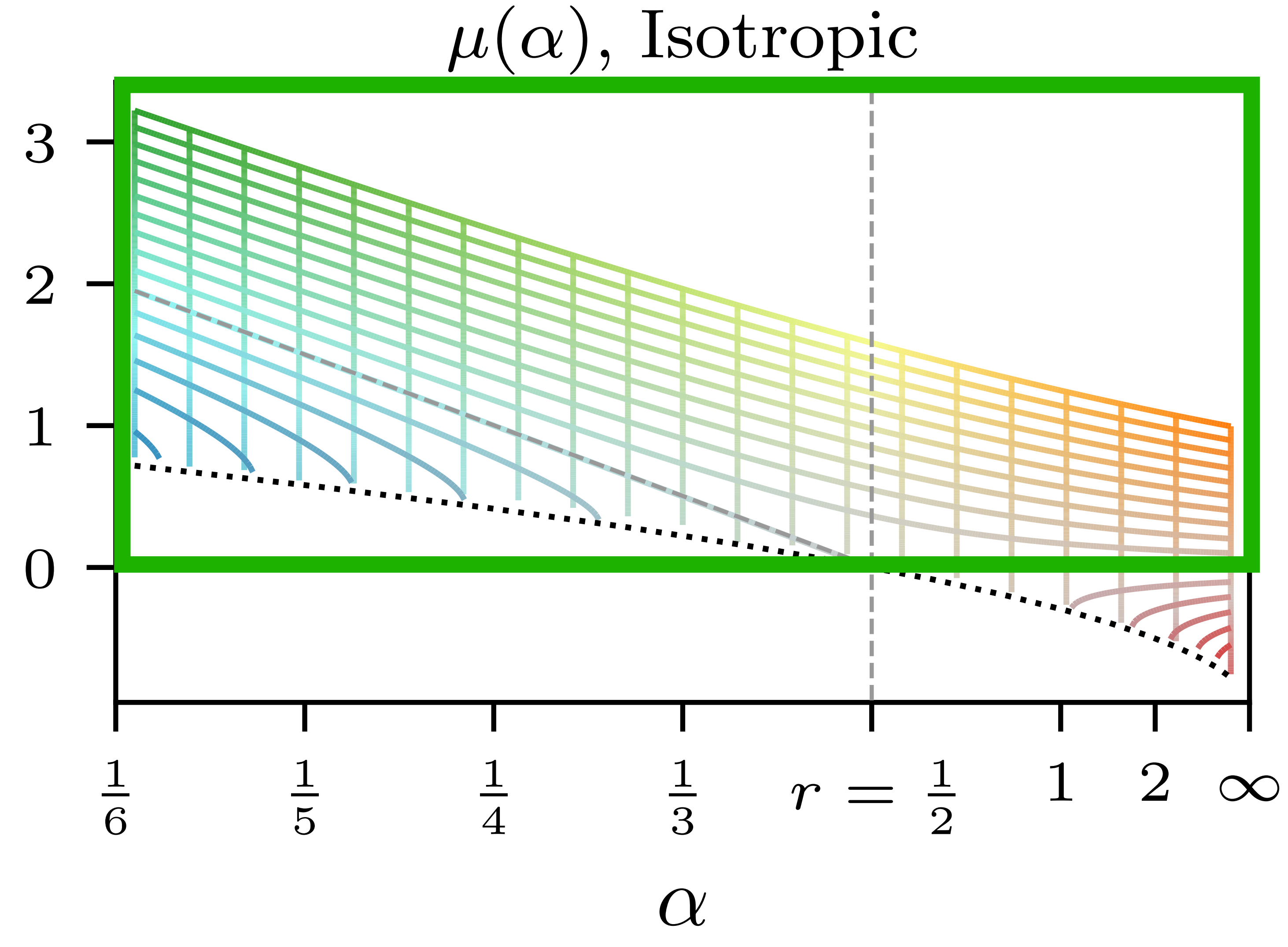

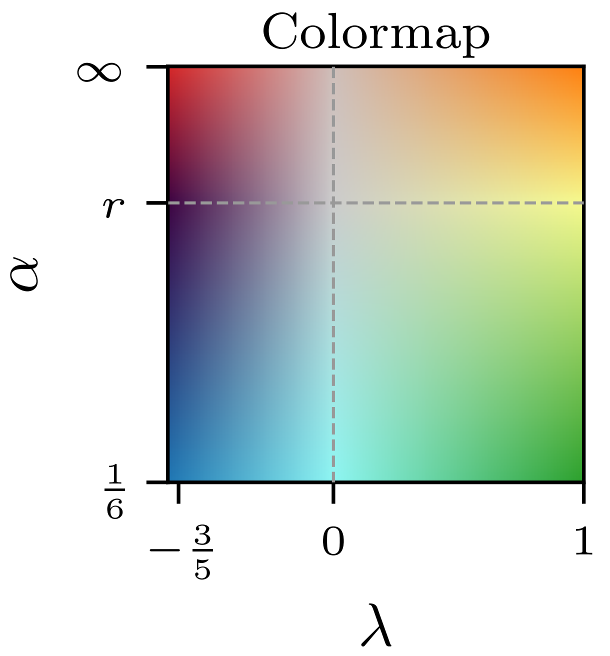

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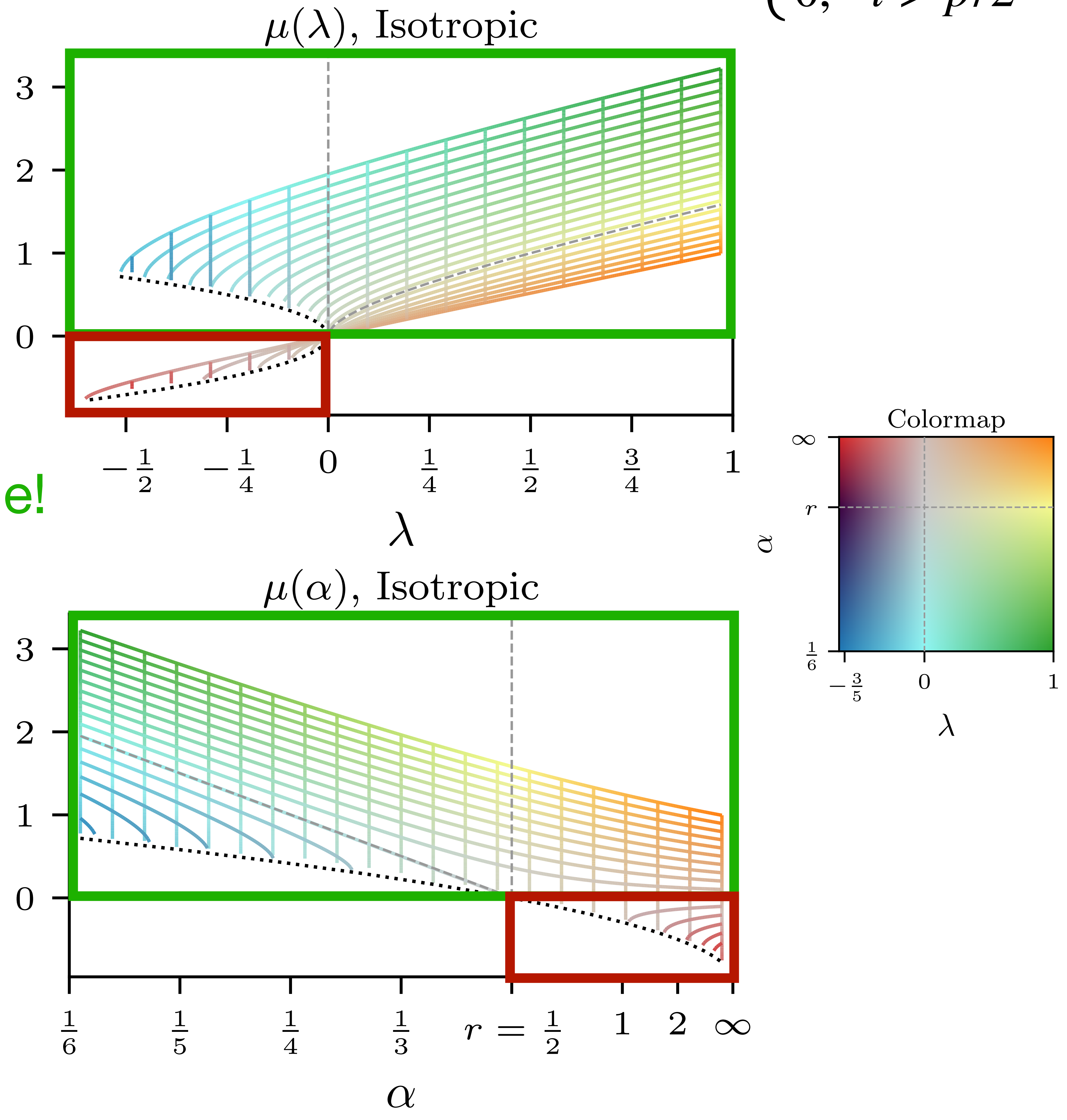

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- **Proof idea:** $\frac{\partial}{\partial z}(\mathbf{A} - z\mathbf{I})^{-1} = (\mathbf{A} - z\mathbf{I})^{-2}$ with carefully placed $\boldsymbol{\Psi}$

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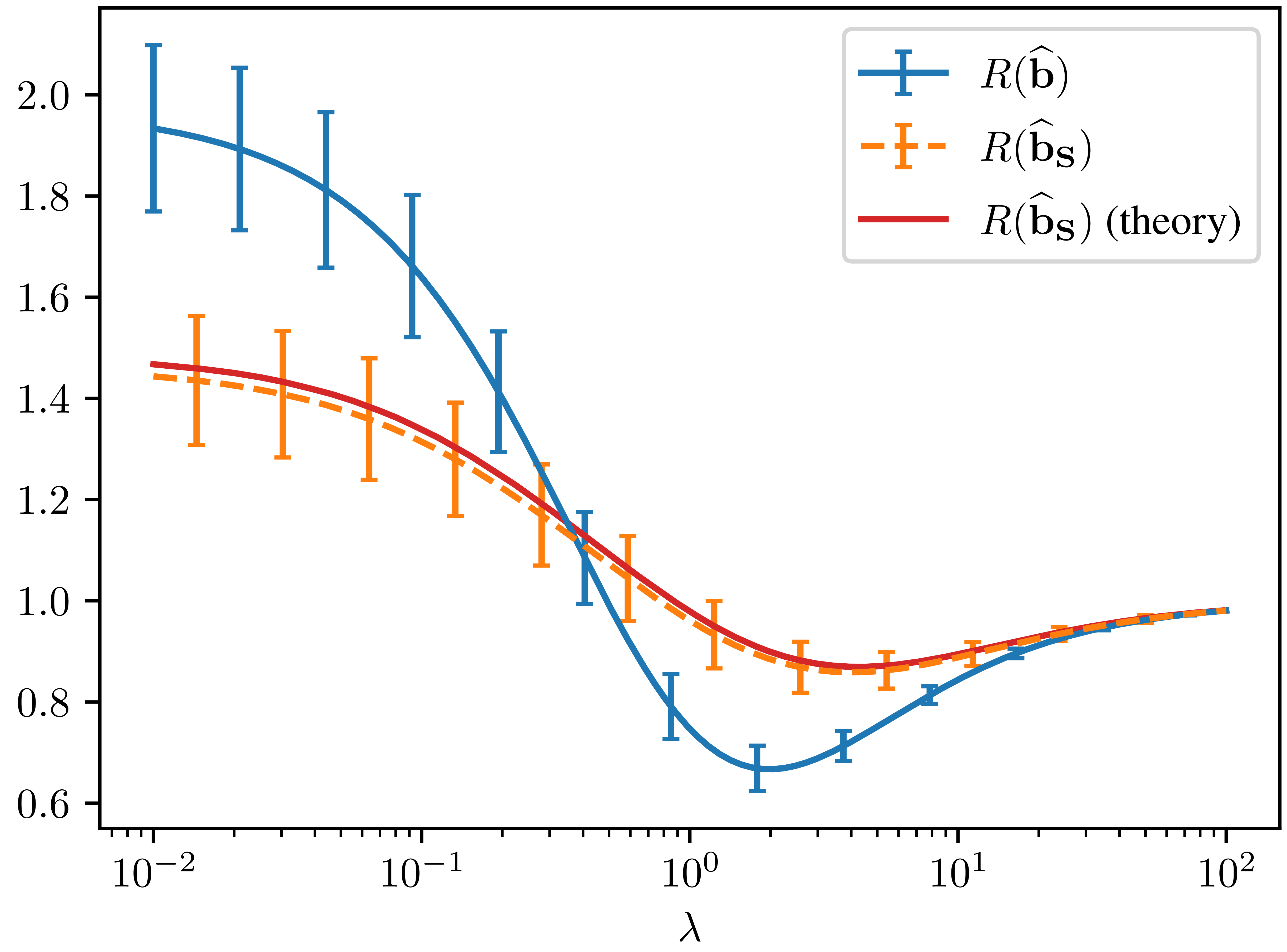

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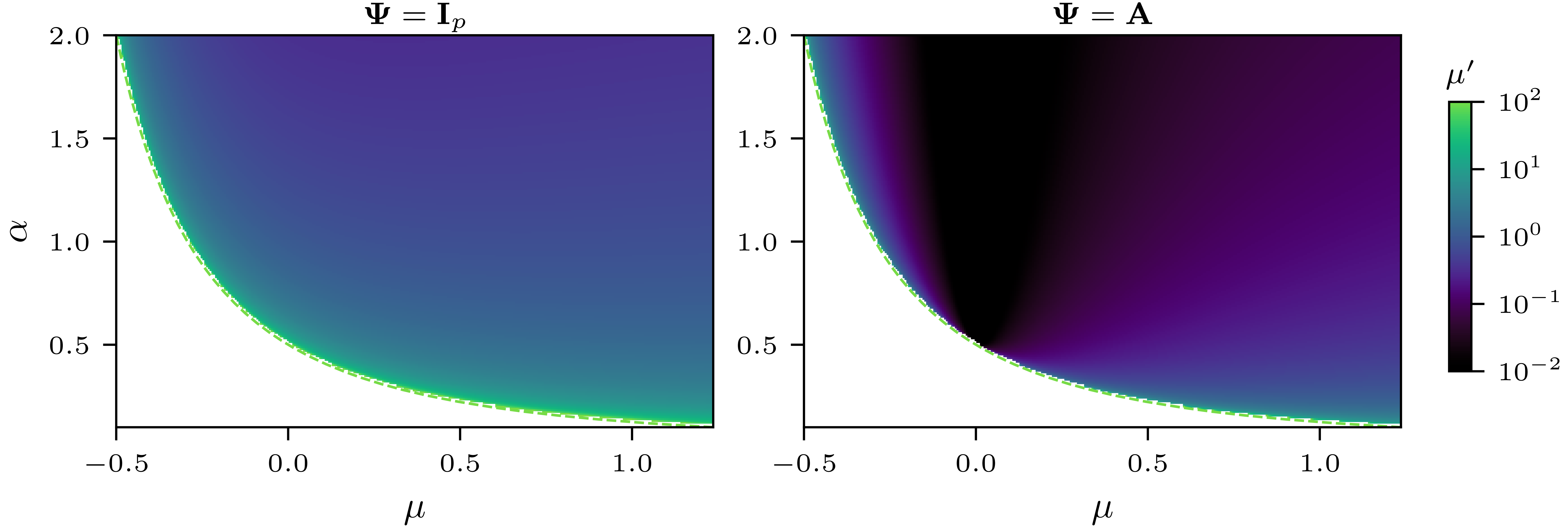

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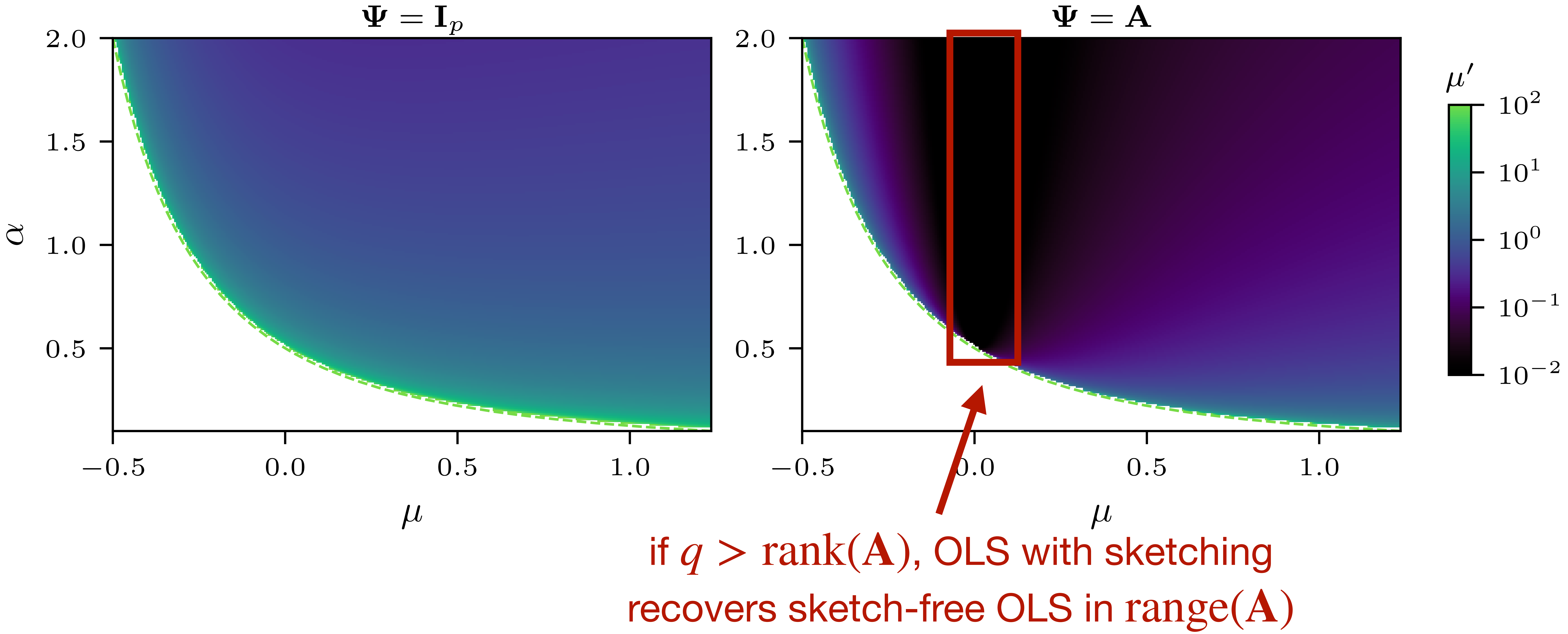

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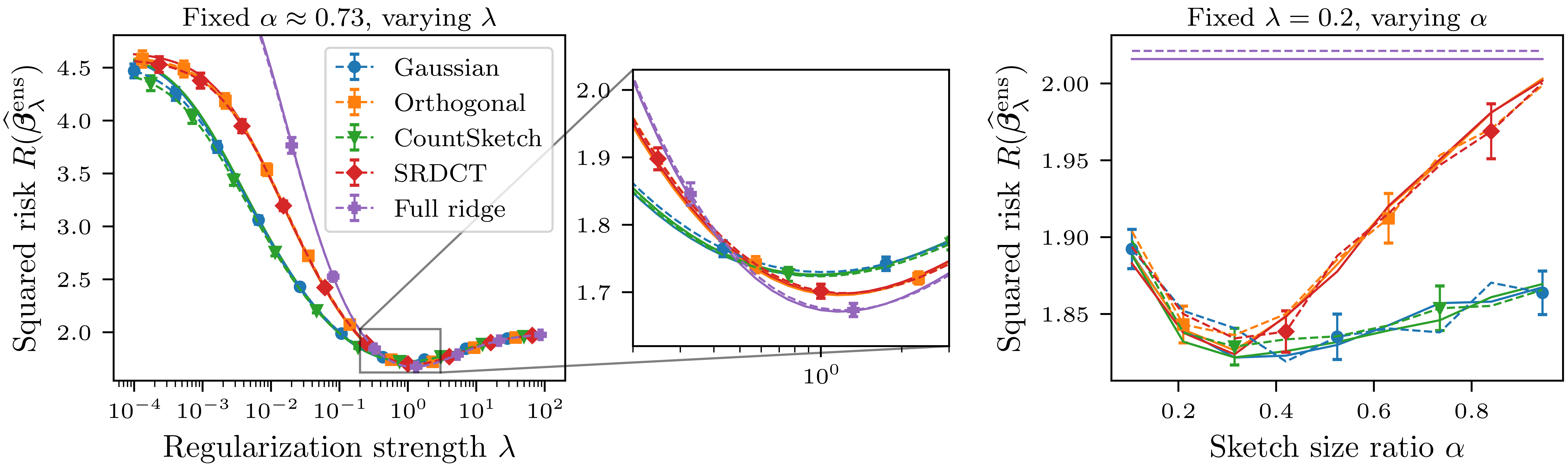

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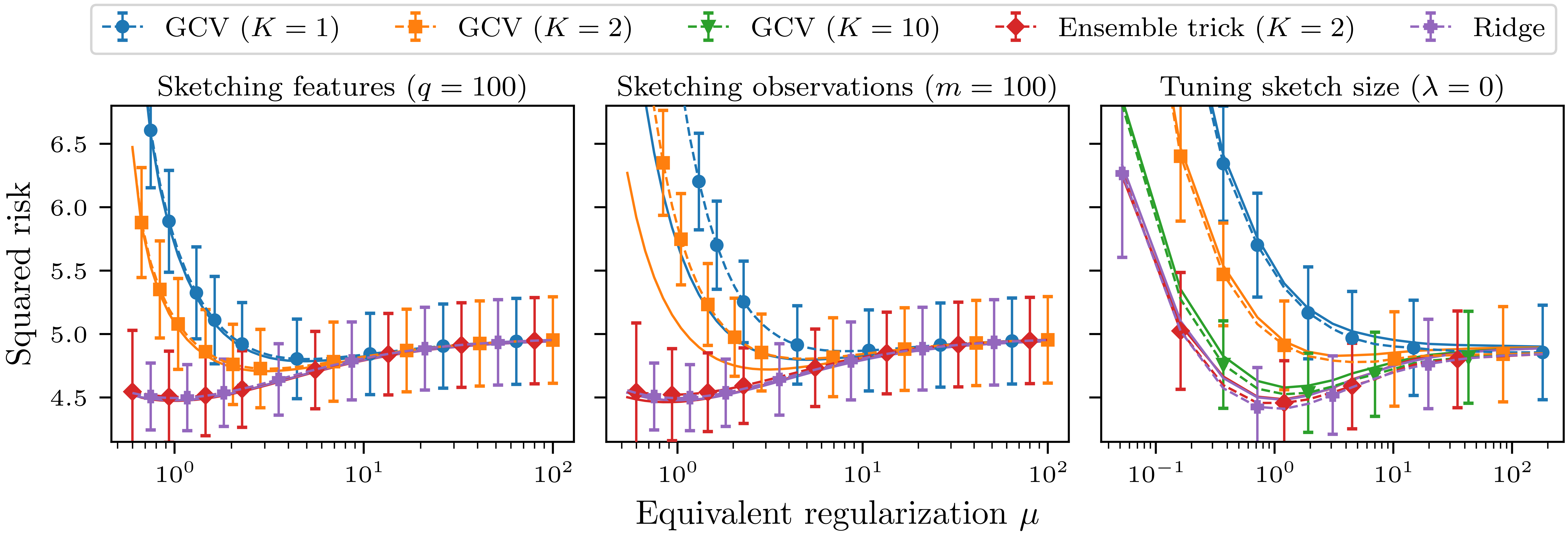

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  - Precise characterization of implicit regularization μ of sketching
    - Covers free sketches $\mathbf{S}$, any data $\mathbf{A}$, sketch ratio α, (negative) λ
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  - Consistency of GCV for sketched ridge regression
    - Efficient risk estimation via ensemble trick
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