1 More sub-Gaussian characterizations

Throughout, $C, c > 0$ denote absolute constants (that may change from line to line), and we write

$$A \lesssim B \quad \text{to mean} \quad A \leqslant CB.$$

Let X be a real-valued random variable, and define the ψ_2 (Orlicz) norm

$$\|X\|_{\psi_2} := \inf\left\{K > 0 : \mathbb{E}\exp\left(\frac{X^2}{K^2}\right) \leqslant 2\right\}.$$

Recall that X is sub-Gaussian if and only if $\|X\|_{\psi_2} < \infty$.

(a) **Sub-Gaussianity via (almost) Gaussian tail domination.** Let $g \sim \mathcal{N}(0,1)$. For two random variables U, V, define the relation

$$U \preceq V \qquad \Longleftrightarrow \qquad \mathbb{P}\{|U| \geqslant t\} \leqslant 2\mathbb{P}\{|V| \geqslant t\} \quad \text{for all } t \geqslant 0.$$

Show that X is sub-Gaussian if and only if there exists $K > 0$ such that

$$X \preceq Kg. \tag{1}$$

More precisely:

(i) If $\|X\|_{\psi_2} \leqslant L$, prove that (1) holds with $K \leqslant CL$.

Solution:
Using Exercise 5 in HW2, $\|X\|_{\psi_2} \leqslant L$ implies:

$$\mathbb{P}\left\{|X| \geqslant t\right\} \leqslant 2\exp\left(-c_1\frac{t^2}{L^2}\right), \quad \text{for all } t \geqslant 0. \tag{2}$$

Using Mill's ratio for all $u > 0$,

$$\mathbb{P}\left\{g \geqslant u\right\} \geqslant \frac{\varphi(u)}{u + u^{-1}}, \qquad \text{where} \qquad \varphi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}.$$

Because g is symmetric around 0, $\mathbb{P}\left\{|g| \geqslant u\right\} = 2\mathbb{P}\left\{g \geqslant u\right\} \geqslant \dfrac{2u}{u^2+1}e^{-u^2/2}$. So,

$$\mathbb{P}\left\{|g| \geqslant u\right\} \geqslant \frac{c_2}{u+1}e^{-u^2/2} \quad \text{for all } u > \frac{1}{2}. \tag{3}$$

Using the crude inequality $\log(u+1) \leqslant u^2/4 + 1$ in (3) gives:

$$\mathbb{P}\left\{|g| \geqslant u\right\} \geqslant c_2 e^{-1}e^{-3u^2/4} \geqslant c_3 e^{-3u^2/4}, \qquad (u > 1/2),$$

for some absolute constant $c_3 > 0$. Let $u = t/K$, which gives

$$2\mathbb{P}\left\{|g| \geqslant \frac{t}{K}\right\} \geqslant 2c_3\exp\left(-\frac{3t^2}{4K^2}\right), \qquad (t > K/2). \tag{4}$$

We need to show $\mathbb{P}\left\{|X| \geqslant t\right\} \leqslant 2\mathbb{P}\left\{|g| \geqslant t/K\right\}$, so using (2) and (4) it suffices to show:

$$\exp\left(-c_1\frac{t^2}{L^2}\right) \leqslant c_3\exp\left(-\frac{3t^2}{4K^2}\right) \Longleftrightarrow \exp\left(-t^2\left(\frac{c_1}{L^2} - \frac{3}{4K^2} \geqslant 0\right)\right) \leq c_3. \tag{5}$$

First choose $K = CL$ such that the coefficient in (5) is positive, i.e.,

$$\frac{c_1}{L^2} - \frac{3}{4K^2} = \frac{1}{L^2}\left(c_1 - \frac{3}{4C^2}\right) \geqslant 0 \iff C \geqslant \sqrt{\frac{3}{4c_1}}, \tag{6}$$

which is an absolute constant. Now, the exponential in (5) is a decreasing function of t when the coefficient is positive, so it's enough to satisfy the bound on the exponential on the left endpoint, which is for $t = K/2$. Putting $t = K/2$ in (5) gives us the condition:

$$\exp\left(\frac{3}{16} - \frac{c_1 C^2}{4}\right) \leqslant c_3 \iff C^2 \geqslant \frac{3}{4c_1} - \frac{4 \log c_3}{c_1}. \tag{7}$$

Now, if $u \in [0, 1/2]$ then $t/K \leqslant 1/2$. This gives $\mathbb{P}\{|g| \geqslant t/K\} \geqslant \mathbb{P}\{|g| \geqslant 1/2\}$, and $\mathbb{P}\{|g| \geqslant 1/2\} > 1/2$ because for a standard normal, $z_{0.25} = 0.674$. So,

$$\mathbb{P}\{|g| \geqslant t/K\} > 1/2 \iff 2\mathbb{P}\{|g| \geqslant t/K\} > 1.$$

However, for any t, $\mathbb{P}\{|X| \geqslant t\} \leqslant 1$, so the bound $\mathbb{P}\{|X| \geqslant t\} \leqslant 2\mathbb{P}\{|g| \geqslant t/K\}$ holds trivially true. Hence, combining (6) and (7), $X \preceq Kg$ for all $K \geqslant CL$ for absolute constant C given by:

$$C^2 = \max\left\{\frac{3}{4c_1}, \frac{3}{4c_1} - \frac{4 \log c_3}{c_1}\right\}. \quad \square$$

(ii) Conversely, if (1) holds for some K, prove that $\|X\|_{\psi_2} \leqslant CK$.

Solution:
Using the assumption that $X \preceq Kg$ for some $K > 0$,

$$\begin{aligned}
\mathbb{P}\{|X| \geqslant t\} &\leqslant 2\mathbb{P}\{|g| \geqslant t/K\} \\
&\leqslant 2 \cdot 2 \exp\left(-\frac{t^2}{2K^2}\right) \quad [g \text{ is gaussian, so has tail bound.}] \\
&= 2 \exp\left(\log 2 - \frac{t^2}{2K^2}\right) \\
&\leqslant 2 \exp\left(-\frac{t^2}{4K^2}\right),
\end{aligned} \tag{8}$$

where the last inequality holds because $\log 2 - \frac{t^2}{2K^2} \leqslant -\frac{t^2}{4K^2}$ whenever $\frac{t^2}{4K^2} \geqslant \log 2$ and for smaller t, $2\exp\left(-\frac{t^2}{4K^2}\right) \geqslant 2e^{-\log 2} = 1$ while probability is always $\leqslant 1$. Hence, (8) holds for all $t \geqslant 0$, and so X is $(2K^2)$–sub-Gaussian. Now, using the result from Exercise 5 in HW2, there exists an absolute constant $c_1 > 0$ such that

$$\mathbb{E} \exp\left(\frac{X^2}{c_1 \cdot 2K^2}\right) \leqslant 2.$$

Hence, by the definition of ψ_2–norm, $\|X\|_{\psi_2} \leqslant CK$ for some absolute constant $C > 0$.

(b) [Bonus] Show by example that part (a) can fail if the definition of $\preceq$ is strengthened by removing the factor 2, i.e., if one demands $\mathbb{P}\{|U| \geqslant t\} \leqslant \mathbb{P}\{|V| \geqslant t\}$ for all $t \geqslant 0$.

[Bonus] Solution:
Suppose we strengthen the relation by defining

$$X \text{ is sub-Gaussian} \iff \mathbb{P}\{|X| \geqslant t\} \leqslant \mathbb{P}\{|g| \geqslant t/K\}, \quad \text{for all } t \geqslant 0, \tag{9}$$

for some $K > 0$. Let $X = \varepsilon$, where $\varepsilon \in \{-1, +1\}$ be a Rademacher random variable. We know, X is sub-Gaussian. Now, for any $t \in (0, 1]$, $\mathbb{P}\{|X| \geqslant t\} = 1$ because $|X| = 1$ a.s. Pick $t = 1/2$, then for any $K > 0$, $\mathbb{P}\left\{|g| \geqslant \frac{1}{2K}\right\} < 1$ because a $1D$–Gaussian has full support on $\mathbb{R}$ and so $\mathbb{P}\{|g| \geqslant u\} < 1$ for every $u > 0$. Hence,

$$\mathbb{P}\left\{|X| \geq \frac{1}{2}\right\} = 1 \nleqslant \mathbb{P}\left\{|g| \geq \frac{1}{2K}\right\},$$

for any finite $K > 0$, and defition (9) fails. Now, if we let the original relation:

$$X \text{ is sub-Gaussian} \iff \mathbb{P}\{|X| \geqslant t\} \leqslant 2\mathbb{P}\{|g| \geqslant t/K\}, \quad \text{for all } t \geqslant 0, \tag{10}$$

for some $K > 0$, then we only need to check it at $t = 1$. We need, for $t = 1$,

$$\mathbb{P}\{|X| \geqslant 1\} = 1 \leqslant 2\mathbb{P}\left\{|g| \geqslant \frac{1}{K}\right\},$$

which can be achieved by choosing K such that $\mathbb{P}\left\{|g| \geqslant \frac{1}{K}\right\} \geqslant \frac{1}{2}$, and (10) holds. □

(c) **[Bonus] A local MGF of X^2 characterization.** Show that X is sub-Gaussian if and only if there exists $K > 0$ such that

$$\mathbb{E}\exp\left(\lambda^2 X^2\right) \leqslant \exp\left(\lambda^2 K^2\right) \quad \text{for all } |\lambda| \leqslant \frac{1}{K}. \tag{11}$$

More precisely:

(i) Assume $\|X\|_{\psi_2} \leqslant L$. Prove that (11) holds with $K \leqslant CL$.

[Bonus] Solution:
Define $U := \frac{X^2}{L^2}$, then using $\|X\|_{\psi_2} \leqslant L$, $\mathbb{E}e^U = \mathbb{E}\exp\left(\frac{X^2}{L^2}\right) \leqslant 2$. Let $\theta = \lambda^2 L^2$ so that $\theta \leqslant 1$ when $|\lambda| \leqslant 1/L$. For $\theta \in [0, 1]$, $x \mapsto x^\theta$ is concave on $\mathbb{R}^+$, so using Jensen's inequality for concave functions,

$$\mathbb{E}\exp\left(\lambda^2 X^2\right) = \mathbb{E}e^{U\theta} = \mathbb{E}\left(e^U\right)^\theta \leqslant \left(\mathbb{E}e^U\right)^\theta \leqslant 2^\theta = \exp\left((\log 2)\lambda^2 L^2\right) \leqslant . \tag{12}$$

Now, to prove (11), it suffices to choose K such that

$$(\log 2)\lambda^2 L^2 \leqslant \lambda^2 K^2 \iff K \geqslant \sqrt{\log 2}\, L.$$

We also need $\{\lambda : |\lambda| \leqslant 1/K\} \subseteq \{\lambda : |\lambda| \leqslant 1/L\}$ so we can use (12), which holds if $K \geqslant L$.

Thus, choose $K = L$ as $\log 2 < 1$. Then,

$$\mathbb{E}\exp\left(\lambda^2 X^2\right) \leqslant \exp\left((\log 2)\lambda^2 L^2\right) \leqslant \exp\left(\lambda^2 L^2\right) = \exp\left(\lambda^2 K^2\right), \quad \text{for all } |\lambda| \leqslant \frac{1}{K}.$$

Hence, (11) holds with $K = L \leqslant CL$ for $C = 1$, and also holds for all $K' \geqslant K$. □

(ii) Conversely, assume (11) holds for some $K > 0$. Prove that X is sub-Gaussian with $\|X\|_{\psi_2} \leqslant 2K$.

[Bonus] Solution:
Here, (11) holds for some $K > 0$, i.e, for some $K > 0$,

$$\mathbb{E}\exp\left(\lambda^2 X^2\right) \leqslant \exp\left(\lambda^2 K^2\right) \quad \text{for all } |\lambda| \leqslant \frac{1}{K}.$$

Substitute $\lambda = \frac{1}{2K} \leqslant \frac{1}{K}$, which gives

$$\mathbb{E}\exp\left(\frac{X^2}{(2K)^2}\right) \leqslant \exp\left(\frac{1}{4K^2} \cdot K^2\right) = \exp\left(\frac{1}{4}\right) < 2.$$

Hence, by the definition of ψ_2−norm, $\|X\|_{\psi_2} \leqslant 2K$ and X is sub-Gaussian. □

2 Sub-exponential characterizations

Let Z be a real-valued random variable with $\mathbb{E}Z = 0$. Define the centered log-MGF

$$\psi_Z(\lambda) := \log \mathbb{E}e^{\lambda Z} \in (-\infty, \infty] \quad (\lambda \in \mathbb{R}),$$

and define the ψ_1 (Orlicz) norm

$$\|X\|_{\psi_1} := \inf\left\{K > 0 : \mathbb{E}\exp\left(\frac{|Z|}{K}\right) \leqslant 2\right\}.$$

(a) **Four sub-exponential properties.** Fix $K > 0$. Consider the statements:

(i) Tail bound: For all $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-c\frac{t}{K}\right).$$

(ii) Moment growth: There exists an absolute constant $C > 0$ such that for all $p \geqslant 1$,

$$(\mathbb{E}|Z|^p)^{1/p} \leqslant CKp.$$

(iii) ψ_1-integrability:

$$\mathbb{E}\exp\left(\frac{|Z|}{K}\right) \leqslant 2.$$

(iv) Local quadratic log-MGF: There exist absolute constants $c, C > 0$ such that for all $|\lambda| \leqslant \frac{c}{K}$,

$$\psi_Z(\lambda) \leqslant C\lambda^2 K^2.$$

Our goal is to show that (i)–(iv) are equivalent up to universal constants (i.e., the parameter K may change by at most an absolute constant factor from one statement to another).

(a1) (i) $\implies$ (ii).

Solution:
From Exercise 4 (a2), in HW1, for a non-negative random variable X, if there exist constants $c, C > 0$ and $\alpha > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\{X \geqslant t\} \leqslant C\exp(-ct^\alpha),$$

then there exists a constant $C' = C'(C, \alpha)$ such that

$$\|X\|_{L_p} = (\mathbb{E}X^p)^{1/p} \leqslant C'(p/c)^{1/\alpha}, \quad \text{for all } p \geqslant 1. \tag{13}$$

For $X = |Z|$, $\alpha = 1$, and $c = 1/K$, using (13), the tail bound,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-c\frac{t}{K}\right),$$

for all $t \geqslant 0$ implies that there exists an absolute constant $C > 0$ such that

$$(\mathbb{E}|Z|^p)^{1/p} \leqslant CKp. \quad \text{for all } p \geqslant 1. \quad \square \tag{14}$$

(a2) (ii) $\implies$ (iii).

Solution:
Using the power series expansion of exponential, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, gives,

$$\mathbb{E}\exp\left(\frac{|Z|}{K}\right) = 1 + \sum_{n=1}^{\infty} \frac{\mathbb{E}|Z|^n}{K^n n!} \leqslant 1 + \sum_{n=1}^{\infty} \frac{\mathbb{E}|Z|^n}{K^n (n/e)^n} \qquad [\because n! \geqslant (n/e)^n]$$
$$\leqslant 1 + \sum_{n=1}^{\infty} \frac{C^n K_1^n n^n}{K^n (n/e)^n} \qquad [\because \text{ moment bound (14)}]$$
$$= 1 + \sum_{n=1}^{\infty} \left(\frac{eCK_1}{K}\right)^n.$$

Now, if we choose $K \geqslant 2eCK_1$, then

$$\mathbb{E}\exp\left(\frac{|Z|}{K}\right) \leqslant 1 + \sum_{n=1}^{\infty} 2^{-n} = 1 + 1 = 2.$$

Hence, moment growth in (ii) implies ψ_1−integrability, i.e.,

$$\mathbb{E}\exp\left(\frac{|Z|}{K}\right) \leqslant 2. \quad \square \tag{15}$$

(a3) (iii) $\implies$ (iv).

Solution:
Using the Taylor expansion, $e^x = 1 + x + \sum_{m=2}^{\infty} \frac{x^m}{m!}$, for $x = \lambda Z$ gives $e^{\lambda Z} = 1 + \lambda Z + \sum_{m=2}^{\infty} \frac{\lambda^m Z^m}{m!}$. Taking expectation and using $\mathbb{E}Z = 0$ gives

$$\mathbb{E}e^{\lambda Z} = 1 + \sum_{m=2}^{\infty} \frac{\lambda^m \mathbb{E}[Z^m]}{m!} \leqslant 1 + \sum_{m=2}^{\infty} \frac{|\lambda|^m \mathbb{E}[|Z|^m]}{m!}, \tag{16}$$

using triangle inequality. Assuming the ψ_1−integrability, i.e., using (15),

$$\mathbb{E}\exp\left(\frac{|Z|}{K}\right) = \sum_{m=0}^{\infty} \frac{\mathbb{E}[|Z|^m]}{K^m m!} \leqslant 2,$$

where all the terms in the sum are non-negative. So, for each m,

$$\frac{\mathbb{E}[|Z|^m]}{K^m m!} \leqslant 2 \implies \mathbb{E}[|Z|^m] \leqslant 2K^m m!.$$

Putting this back in (16) gives:

$$\mathbb{E}e^{\lambda Z} \leqslant 1 + \sum_{m=2}^{\infty} \frac{|\lambda|^m}{m!} \cdot 2K^m m! = 1 + 2\sum_{m=2}^{\infty} (|\lambda| K)^m, \tag{17}$$

where the latter geometric sum converges if $|\lambda| K < 1$. Now, for $|\lambda| \leqslant \frac{1}{2K}$, $1 - 2|\lambda|K \geqslant 1/2$, so (17) becomes,

$$\mathbb{E}e^{\lambda Z} \leqslant 1 + \frac{2\lambda^2 K^2}{1 - 2|\lambda| K} \leqslant 1 + 4\lambda^2 K^2.$$

Now, taking log and using the fact $\log(1+u) \leqslant u$ for $u \geqslant 0$,

$$\log \mathbb{E}e^{\lambda Z} \leqslant \log(1 + 4\lambda^2 K^2) \leqslant 4\lambda^2 K^2.$$

Hence, there exists absolute constants $c, C > 0$ such that for all $|\lambda| \leqslant \frac{c}{K}$,

$$\psi_Z(\lambda) = \log \mathbb{E}e^{\lambda Z} \leqslant C\lambda^2 K^2. \quad \square \tag{18}$$

(a4) Assume (iv). Show there exists an absolute constant $c > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-c \min\left(\frac{t^2}{k^2}, \frac{t}{K}\right)\right).$$

Deduce in particular (i) (after adjusting constants).

Solution:
Using Chernoff bound for the upper tail,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \inf_{\lambda > 0} \exp\left(-\lambda t + \psi_Z(\lambda)\right) \leqslant \exp\left(-\lambda t + C_0 \lambda^2 K^2\right),$$

for all $\lambda \in [0, c_0/K]$, where the latter inequality is true assuming (iv), i.e., local quadratic log-MGF. Let $f(\lambda) = -\lambda t + C_0\lambda^2 K^2 \implies f'(\lambda) = -t + 2C_0\lambda K^2$. Setting $f' = 0$ gives

$$-t + 2C_0\lambda K^2 = 0 \implies \lambda^* = \frac{t}{2C_0 K^2}.$$

Now, we have two possible cases:

- $\lambda^* \leqslant c_0/K$, which happens iff $t \leqslant 2c_0 C_0 K$. So, for $t \leqslant 2c_0 C_0 K$ the unconstrained optimizer is admissible, and plugging this λ^* into $f(\lambda)$ gives:

$$f(\lambda^*) = -\frac{t^2}{2C_0K^2} + C_0K^2 \cdot \frac{t^2}{4C_0^2K^4} = -\frac{t^2}{2C_0K^2} + \frac{t^2}{4C_0K^2} = -\frac{t^2}{4C_0K^2}.$$

  So, for $t \lesssim K$ we get Gaussian-type tail, i.e.,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \exp\left(-\frac{t^2}{4C_0K^2}\right). \tag{19}$$

- $\lambda^* > c_0/K$, i.e. the unconstrained optimizer is outside the feasible region $[0, c_0/K]$, which happens iff $t > 2c_0C_0K$. Here, we use the endpoint, $\lambda^{**} = c_0/K$, which gives

$$f(\lambda^{**}) = -\frac{c_0 t}{K} + c_0^2 C_0 \leqslant -\frac{c_0 t}{K} + \frac{c_0 t}{2K} = -\frac{c_0}{2} \cdot \frac{t}{K},$$

  where the inequality is true because for $t > 2c_0C_0K \implies \frac{c_0 t}{2K} \geqslant c_0^2 C_0$. So, for $t \gtrsim K$ we get Exponential-type tail, i.e.,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \exp\left(-\frac{c_0}{2} \cdot \frac{t}{K}\right). \tag{20}$$

Combining (19) and (20), there exists an absolute constant $c > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \exp\left(-c \min\left(\frac{t^2}{k^2}, \frac{t}{K}\right)\right). \tag{21}$$

Now, we apply the same argument to $-Z$. Since $\psi_{-Z}(\lambda) = \psi_Z(-\lambda)$, assumption (iv) also holds for $-Z$, so

$$\mathbb{P}\{-Z \geqslant t\} \leqslant \exp\left(-c \min\left(\frac{t^2}{k^2}, \frac{t}{K}\right)\right). \tag{22}$$

Finally, using (21) and (22), and union bound gives:

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant \mathbb{P}\{Z \geqslant t\} + \mathbb{P}\{-Z \geqslant t\} = 2\exp\left(-c \min\left(\frac{t^2}{k^2}, \frac{t}{K}\right)\right). \quad \square \tag{23}$$

Now, for $t \geqslant K$, $\min\left(\frac{t^2}{k^2}, \frac{t}{K}\right) \geqslant \frac{t}{2K}$, so (23) gives

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-c\frac{t}{2K}\right). \tag{24}$$

Similarly, when $0 \leqslant t \leqslant K$, we know $2\exp\left(-\frac{t}{\alpha K}\right) \geqslant 2\exp\left(-\frac{1}{\alpha}\right)$, where

$$2\exp\left(-\frac{1}{\alpha}\right) \geqslant 1 \iff \alpha \geqslant \frac{1}{\log 2}.$$

Thus, for $\alpha \geqslant \frac{1}{\log 2}$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 1 \leqslant 2\exp\left(-\frac{t}{\alpha K}\right),. \tag{25}$$

Let $C' = \min\left\{\frac{c}{2}, \frac{1}{\alpha}\right\}$, then combining (24) and (25), for all $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-C'\frac{t}{K}\right). \quad \square \tag{26}$$

(b) **[Bonus] Comparison with sub-Gaussianity.** Show that sub-Gaussian implies sub-exponential: if Z is σ^2–sub-Gaussian, then $\|Z\|_{\psi_1} \leqslant C\sigma$. Give an explicit example of a centered random variable that is sub-exponential but not sub-Gaussian (e.g. $Z = X - \mathbb{E}X$ with $X \sim \text{Exp}(1)$), and justify both claims from tails.

[Bonus] Solution:
Assume Z is (σ^2)–sub-Gaussian, then

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{t^2}{2\sigma^2}\right), \quad \forall t \geqslant 0.$$

For any $t \geqslant 0$, $\frac{t^2}{2\sigma^2} \geqslant \frac{t}{2\sigma}$ whenever $t \geqslant \sigma$. So, for all $t \geqslant \sigma$, $\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{t}{2\sigma}\right)$.

And, for $0 \leqslant t \leqslant \sigma$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 1 \leqslant 2\exp\left(-\frac{t}{2\sigma}\right),$$

because $e^{-t/(2\sigma)} \geqslant e^{-1/2} > 1$. Hence, for all $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{t}{2\sigma}\right),$$

i.e., Z is sub-exponential and $\|Z\|_{\psi_1} \leqslant C\sigma$ for some constant C. Thus,

$$Z \text{ is sub-Gaussian} \implies Z \text{ is sub-exponential.} \quad \square$$

Now, let $Z = X - 1$ where $X \sim \text{Exp}(1)$, then $\mathbb{E}Z = 0$ (i.e., Z is centered). Then, for $t \geqslant 0$,

$$\mathbb{P}\{Z \geqslant t\} = \mathbb{P}\{Z \geqslant 1 + t\} = e^{-(1+t)} = e^{-1}e^{-t} \leqslant e^{-t}.$$

We know, $Z \geqslant -1$, so for $t > 1$, $\mathbb{P}\{Z \leqslant -t\} = 0$. Therefore, using the union bound gives

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2e^{-t}, \quad (t \geqslant 1).$$

After adjusting for constants,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2e^{-ct}, \quad (t \geqslant 0),$$

so Z has exponential tails, hence Z is sub-exponential. Now, for Z to be sub-Gaussian, there need to exist some $C > 0$ such that

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{t^2}{2C^2}\right), \quad (t \geqslant 0). \tag{27}$$

But for $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \geqslant \mathbb{P}\{Z \geqslant t\} = e^{-(t+1)}.$$

So, (27) would force $\exp(-(t+1)) \leqslant 2\exp\left(-\frac{t^2}{2C^2}\right)$ for all $t \geq 0$. Taking logs:

$$-(t+1) \leqslant \log 2 - \frac{t^2}{2C^2} \implies t \geqslant \frac{t^2}{2C^2} - \log 2 - 1,$$

which is not possible for large t because the right side is quadratic in t. So, there does not exist any $C > 0$ such that (27) holds for all $t \geqslant 0$, hence Z is not sub-Gaussian. $\square$

3 Bounding ℓ_p norms of random vectors

Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$. For $p \in [1, \infty)$, define

$$\|X\|_p := \left(\sum_{i=1}^{n} |X_i|^p\right)^{1/p}, \quad \|X\|_\infty := \max_{1 \leqslant i \leqslant n} |X_i|.$$

Throughout, you may freely use the standard sub-Gaussian consequences: there exist universal constants $c, C > 0$ such that for every i and every $q \geqslant 1$,

$$(\mathbb{E}|X_i|^q)^{1/q} \leqslant CK\sqrt{q}, \quad \text{and} \quad \mathbb{P}\{|X_i| \geqslant t\} \leqslant 2\exp\left(-c\frac{t^2}{K^2}\right) \quad (t \geqslant 0).$$

(a) **Expected ℓ_p norm (finite p).** Show that for every $p \in [1, \infty)$,

$$\mathbb{E}\|X\|_p \leqslant CK\sqrt{p}\, n^{1/p}. \tag{28}$$

Solution:
$x \mapsto x^{1/p}$ for $p \geqslant 1$ is concave, so using Jensen's inequality for concave function gives:

$$\mathbb{E}\|X\|_p = \mathbb{E}\left[\left(\sum_{i=1}^{n} |X_i|^p\right)^{1/p}\right] \leqslant \left(\mathbb{E}\left[\sum_{i=1}^{n} |X_i|^p\right]\right)^{1/p} = \left(nC^pK^pp^{p/2}\right)^{1/p} = CK\sqrt{p}\, n^{1/p},$$

where we use the identical moment bound for each X_i. □

(b) **Expected ℓ_∞ norm.** Prove that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_\infty \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \leqslant 2e^{-ct^2}.$$

Deduce that

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n}.$$

Solution:
Because $\max_i |X_i| \geqslant u \implies |X_i| \geqslant u$ for all i, using union bound gives:

$$\begin{aligned}
\mathbb{P}\left\{\|X\|_\infty \geqslant CK\left(\sqrt{\log n} + t\right)\right\} &\leqslant \sum_{i=1}^{n} \mathbb{P}\left\{|X_i| \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \\
&\leqslant n \cdot 2\exp\left(c_1C^2(\sqrt{\log n} + t)^2\right) && [\text{tail bound for } X_i] \\
&\leqslant 2\exp\left(-c_1C^2(\log n + t^2) + \log n\right) && [(a+b)^2 \leqslant a^2 + b^2] \\
&= 2\exp\left((1 - c_1C^2)\log n\right) \cdot \exp\left(-c_1Ct^2\right).
\end{aligned}$$

Here, if we choose $1 - c_1C^2 \leqslant 0 \iff C \geqslant 1/\sqrt{c_1}$, then $\exp\left((1 - c_1C^2)\log n\right) \leqslant 1$. Hence, there exists universal constants $C \geqslant 1/\sqrt{c_1} > 0$ and $C = c_1C$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_\infty \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \leqslant 2e^{-ct^2}. \quad \square \tag{29}$$

Now, using the tail integral identity $\mathbb{E}Y = \int_0^\infty \mathbb{P}\{Y \geqslant s\}\, ds$ gives:

$$\mathbb{E}\|X\|_\infty = \int_0^\infty \mathbb{P}\{\|X\|_\infty \geqslant s\}\, ds = \int_0^{CK\sqrt{\log n}} \mathbb{P}\{\|X\|_\infty \geqslant s\}\, ds + \int_{CK\sqrt{\log n}}^\infty \mathbb{P}\{\|X\|_\infty \geqslant s\}\, ds,$$

where we can bound the first integral by $CK\sqrt{\log n}$ because probability is always $\leqslant 1$. So,

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n} + \int_{CK\sqrt{\log n}}^\infty \mathbb{P}\{\|X\|_\infty \geqslant s\}\, ds.$$

Let $s = CK\left(\sqrt{\log n} + t\right) \implies ds = CK\, dt$, then using the tail bound in (29):

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n} + CK\int_{CK\sqrt{\log n}}^\infty 2e^{-ct^2} dt \leqslant CK\sqrt{\log n} + 2CK\int_0^\infty e^{-ct^2} dt,$$

where the latter integral is just a constant, as it's an integral of a Gaussian kernel. Then,

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n} + C_1K \leqslant C'K\sqrt{\log n}.$$

Hence, there exists universal constant $C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n}. \tag{30}$$

(c) **Two regimes of expected ℓ_p norms.** Show that

$$\mathbb{E}\|X\|_p \leqslant \begin{cases} CK\sqrt{p}\, n^{1/p}, & 1 \leqslant p \leqslant \log n, \\ CK\sqrt{\log n}, & p \geqslant \log n, \end{cases}$$

up to universal constants.

Solution:
From Exercise 1 in HW1, we have for every $p \geqslant 1$, $\|X\|_p \leqslant n^{1/p}\|X\|_\infty$. So, using (30):

$$\mathbb{E}\|X\|_p \leqslant n^{1/p}\|X\|_\infty \leqslant CK\sqrt{\log n}\, n^{1/p}. \tag{31}$$

Combining (28) and (31) gives:

$$\mathbb{E}\|X\|_p \leqslant \min\left\{CK\sqrt{p}\, n^{1/p}, CK\sqrt{\log n}\, n^{1/p}\right\}.$$

When $p \geqslant \log n, n^{1/p} \leqslant e$ so we can merge $n^{1/p}$ into C in (31) for $p \geqslant \log n$. Hence,

$$\mathbb{E}\|X\|_p \leqslant \begin{cases} CK\sqrt{p}\, n^{1/p}, & 1 \leqslant p \leqslant \log n, \\ CK\sqrt{\log n}, & p \geqslant \log n, \end{cases}$$

up to universal constants. □

(d) **[Bonus] High-probability bound for ℓ_p norm when $p \geqslant \log n$.** Fix $p \in [\log n, \infty]$. Show that there exists universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_p \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \leqslant 2e^{-ct^2}.$$

[Bonus] Solution:
Similar to part (c), when $p \geqslant \log n$, $n^{1/p} \leqslant e$ so we get $\|X\|_p \leqslant e\|X\|_\infty$. Fix $p \in [\log n, \infty]$. Using (29), there exists a universal constants $c, C_0 > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_\infty \geqslant C_0 K\left(\sqrt{\log n} + t\right)\right\} \leqslant 2e^{-ct^2}. \tag{32}$$

Choose $C = eC_0$, then

$$\|X\|_p \geqslant CK\left(\sqrt{\log n} + t\right) \implies \|X\|_\infty \geqslant \frac{\|X\|_p}{e} \geqslant C_0 K\left(\sqrt{\log n} + t\right).$$

This gives the event $A := \left\{\|X\|_\infty \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \subseteq \left\{\|X\|_\infty \geqslant C_0 K\left(\sqrt{\log n} + t\right)\right\} =: B$, i.e. $\mathbb{P}\{A\} \leqslant \mathbb{P}\{B\}$. Using (32), for $p \in [\log n, \infty]$, there exists $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_p \geqslant CK\left(\sqrt{\log n} + t\right)\right\} \leqslant 2e^{-ct^2}. \quad \square \tag{33}$$

(e) **[Bonus] High-probability bound for ℓ_p norm when $p \leqslant \log n$.** Fix $p \in [1, \log n]$. Show that for every $q \geqslant p$,

$$\left(\mathbb{E}\|X\|_p^q\right)^{1/q} \leqslant CK\, n^{1/p}\sqrt{q}.$$

Deduce that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_p \geqslant CK\, n^{1/p}\,(\sqrt{p} + t)\right\} \leqslant 2e^{-ct^2}.$$

[Bonus] Solution:
Fix $p \in [1, \log n]$ and define $Y := \|X\|_p$. Let $r = q/p \geqslant 1$, then using the identity $(\sum a_i)^r \leqslant n^{r-1}\sum a_i^r$ gives $Y^q \leqslant n^{q/p-1}\sum_{i=1}^n |X_i|^q$. So,

$$\mathbb{E}Y^q \leqslant n^{q/p-1}\sum_{i=1}^n \mathbb{E}|X_i|^q \leqslant n^{q/p-1} n (C_1 K\sqrt{q})^q = \left(C_1 K\, n^{1/p}\sqrt{q}\right)^q,$$

where we use sub-Gaussian moment bound for each X_i. Hence, for every $q \geqslant p$,

$$\left(\mathbb{E}\|X\|_p^q\right)^{1/q} \leqslant C_1 K\, n^{1/p}\sqrt{q}. \quad \square \tag{34}$$

Let $q = p + t^2$, then using Markov's inequality:

$$\mathbb{P}\left\{Y \geqslant e\,(\mathbb{E}Y^q)^{1/q}\right\} = \mathbb{P}\{Y^q \geqslant e^q\mathbb{E}[Y^q]\} \leqslant e^{-q} = e^{-(p+t^2)} \leqslant e^{-t^2}. \tag{35}$$

Using the bound for $(\mathbb{E}Y^q)^{1/q}$ from (34) in (35) gives,

$$\mathbb{P}\left\{Y \geqslant eC_1 K\, n^{1/p}\sqrt{p + t^2}\right\} \leqslant e^{-t^2} \implies \mathbb{P}\left\{Y \geqslant eC_1 K\, n^{1/p}(\sqrt{p} + t)\right\} \leqslant e^{-t^2},$$

using the fact that $\sqrt{p+t^2} \leqslant \sqrt{p}+t$. Hence, there exists universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\|X\|_p \geqslant CK\, n^{1/p}\left(\sqrt{p}+t\right)\right\} \leqslant 2e^{-ct^2}. \quad \square \tag{36}$$

(f) **[Bonus] Thin shell for $\|X\|_2$ (sharper than part (d)).** Assume in addition that $\mathbb{E}X_i^2 = 1$ for all i. Show that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\left|\|X\|_2 - \sqrt{n}\right| \geqslant t\right\} \leqslant 2\exp\left(-c\frac{t^2}{K^4}\right).$$

[Bonus] Solution:
Here, each X_i is sub-Gaussian with $\|X_i\|_{\psi_2} \leqslant K$. So, X_i^2 will be sub-exponential with $\left\|X_i^2\right\|_{\psi_1} \leqslant C_1\|X_i\|_{\psi_2}^2 \leqslant C_1K^2$, for some constant $C_1 > 0$. Let $Y_i = X_i^2 - 1$, then Y_i will also be sub-exponential with $\|Y_i\|_{\psi_1} \leqslant C_1K^2$ and Y_i will be centered as $\mathbb{E}X_i^2 = 1$ (from assumption). Define $Z := n^{-1}\|X\|_2^2 - 1 = n^{-1}\sum_{i=1}^n X_i^2 - 1 = n^{-1}\sum_{i=1}^n Y_i$. Here, Y_i's are independent, so using Bernstein's inequality for the sum of independent sub-exponential gives,

$$\mathbb{P}\{|Z| \geqslant u\} \leqslant 2\exp\left\{-c_1 n\cdot\min\left(\frac{u^2}{K^4}, \frac{u}{K^2}\right)\right\}, \tag{37}$$

for all $u \geqslant 0$, where $c_1 > 0$ is an absolute constant. Let $\delta = \frac{t}{\sqrt{n}}$, then using the implication $\left|\frac{\|X\|_2}{\sqrt{n}} - 1\right| \geqslant \delta \implies \left|\frac{\|X\|_2^2}{n} - 1\right| \geqslant \max\left\{\delta, \delta^2\right\}$ gives,

$$\mathbb{P}\left\{\left|\|X\|_2 - \sqrt{n}\right| \geqslant t\right\} \leqslant \mathbb{P}\left\{|Z| \geqslant \max\left(\frac{t}{\sqrt{n}}, \frac{t^2}{n}\right)\right\}. \tag{38}$$

Putting $u = \max\left(\frac{t}{\sqrt{n}}, \frac{t^2}{n}\right)$ in (37), we get

$$\mathbb{P}\left\{\left|\|X\|_2 - \sqrt{n}\right| \geqslant t\right\} \leqslant 2\exp\left\{-c_1 n\cdot\min\left(\frac{u^2}{K^4}, \frac{u}{K^2}\right)\right\}. \tag{39}$$

Now, we have two cases:

i. When $t \leqslant \sqrt{n}$, $u = t/\sqrt{n}$. Then,

$$c_1 n\cdot\frac{u^2}{K^4} = c_1\cdot\frac{t^2}{K^4}.$$

Similarly,

$$c_1 n\cdot\frac{u}{K^2} = c_1\frac{t\sqrt{n}}{K^2} \geqslant c_1\cdot\frac{t^2}{K^2} \geqslant c_1\cdot\frac{t^2}{K^4}.$$

So, in this case:

$$c_1 n\cdot\min\left(\frac{u^2}{K^4}, \frac{u}{K^2}\right) \geqslant c_1\cdot\frac{t^2}{K^4}.$$

ii. When $t \geqslant \sqrt{n}$, $u = t^2/n$. Then,

$$c_1 n \cdot \frac{u^2}{K^4} = c_1 \cdot \frac{t^4}{nK^4} \geqslant c_1 \cdot \frac{t^2}{K^4},$$

because $t^2 \geqslant n$. Similarly,

$$c_1 n \cdot \frac{u}{K^2} = c_1 \cdot \frac{t^2}{K^2} \geqslant c_1 \cdot \frac{t^2}{K^4}.$$

So, in this case:

$$c_1 n \cdot \min\left(\frac{u^2}{K^4}, \frac{u}{K^2}\right) \geqslant c_1 \cdot \frac{t^2}{K^4}.$$

Hence, combining both cases and putting it back in (39), there exists universal constant $c > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{\left|\|X\|_2 - \sqrt{n}\right| \geqslant t\right\} \leqslant 2\exp\left(-c\frac{t^2}{K^4}\right). \quad \square$$

4 Bounding L_p norms of linear forms (Khintchine inequalities)

Throughout, let $X_1, \dots, X_n$ be independent random variables and let $a = (a_1, \dots, a_N) \in \mathbb{R}^n$. Write

$$S(a) := \sum_{i=1}^{N} a_i X_i, \qquad \|a\|_2 := \left(\sum_{i=1}^{N} a_i^2\right)^{1/2}, \qquad \|a\|_\infty := \max_{1\leqslant i\leqslant N} |a_i|.$$

Throughout, $C, c > 0$ denote absolute constants (that may change from line to line).

(a) **Sub-Gaussian Khintchine L_p bound ($p \geqslant$).** Assume

$$\mathbb{E}X_i = 0, \qquad \mathbb{E}X_i^2 = 1, \qquad \|X_i\|_{\psi_2} \leqslant K_2 \quad \text{for all } i,$$

for some $K_2 \geqslant 1$.Show that for every $p \in [2, \infty)$,

$$\|a\|_2 \leqslant \|S(a)\|_{L_p} \leqslant CK_2\sqrt{p}\,\|a\|_2\,. \tag{40}$$

Solution:
Using the monotonicity of L_p–norm, for $0 < p \leqslant q \leqslant \infty$, $\|X\|_{L_p} \leqslant \|X\|_{L_q}$. So, for all $p \in [2, \infty)$,

$$\|S(a)\|_{L_2} \leqslant \|S(a)\|_{L_p},$$

where

$$\|S(a)\|_{L_2}^2 = \mathbb{E}\left(\sum_{i=1}^{N} a_i X_i\right)^2 = \sum_{i=1}^{N} a_i^2 \mathbb{E}[X_i^2] + \sum_{1\leqslant i<j\leqslant N} a_i a_j \mathbb{E}[X_i X_j] = \sum_{i=1}^{N} a_i^2 = \|a\|_2^2,$$

because X_i's are independent, $\mathbb{E}[X_i X_j] = 0$ for all $i \neq j$. So, for all $p \in [2, \infty)$, $\|a\|_2 \leqslant \|S(a)\|_{L_p}$.
Because X_i is Gaussian with $\|X_i\|_{\psi_2} \leqslant K_2$, $a_i X_i$ will also be Gaussian with $\|a_i X_i\|_{\psi_2} \leqslant |a_i| K_2$.
Now, using the sub-Gaussian Hoeffding inequality for sum of independent Gaussians:

$$\|S(a)\|_{\psi_2}^2 \leqslant C_1 \sum_{i=1}^{N} \|a_i X_i\|_{\psi_2}^2 \leqslant C_1 K_2^2 \sum_{i=1}^{N} a_i^2 = C_1 K_2^2 \|a\|_2^2 \implies \|S(a)\|_{\psi_2} \leqslant C_2 K_2 \|a\|_2\,.$$

$S(a)$ is sub-Gaussian because it's a sum of sub-Gaussians, and for a sub-Gaussian Y, the relation between moment growth and exponential integrability gives $\|Y\|_{L_p} \leqslant C_3\sqrt{p}\,\|Y\|_{\psi_2}$.
So, $\|S(a)\|_{L_p} \leqslant C\sqrt{p}K_2 \|a\|_2$, and hence for every $p \in [2, \infty)$,

$$\|a\|_2 \leqslant \|S(a)\|_{L_p} \leqslant CK_2\sqrt{p}\,\|a\|_2\,. \quad \square$$

(b) **Sub-exponential Bernstein-Khintchine tail bound.** Assume instead that

$$\mathbb{E}X_i = 0, \qquad \|X_i\|_{\psi_1} \leqslant K_1 \quad \text{for all } i,$$

for some $K_1 > 0$. Show that there exists an absolute constant $c > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left\{|S(a)| \geqslant t\right\} \leqslant 2\exp\left[-c\min\left(\frac{t^2}{K_1^2\|a\|_2^2}, \frac{t}{K_1\|a\|_\infty}\right)\right]. \tag{41}$$

Solution:
X_i's are sub-Exponential with $\|X_i\|_{\psi_1} \leqslant K_1$, so $\|a_i X_i\|_{\psi_1} \leqslant |a_i| K_1$. And,

$$V = \sum_{i=1}^{N} \|a_i X_i\|_{\psi_1}^2 \leqslant K_1^2 \sum_{i=1}^{N} a_i^2 = K_1^2 \|a\|_2^2, \quad \text{and} \quad B = \max_{1\leqslant i\leqslant N} \|a_i X_i\|_{\psi_1} \leqslant K_1 \max_{1\leqslant i\leqslant N} a_i = K_1 \|a\|_\infty .$$

Using the Bernstein inequality for sub-exponential sums on $S(a)$, $\exists c > 0$ s.t. for all $t \geqslant 0$,

$$\mathbb{P}\{|S(a)| \geqslant t\} \leqslant 2\exp\left[-c\min\left(\frac{t^2}{V}, \frac{t}{B}\right)\right] \leqslant 2\exp\left[-c\min\left(\frac{t^2}{K_1^2\|a\|_2^2}, \frac{t}{K_1\|a\|_\infty}\right)\right]. \quad \square$$

(c) **Sub-exponential Bernstein-Khintchine L_p bound ($p \geqslant 2$).** Use (41) to show that there exists an absolute constant $C > 0$ such that for every $p \geqslant 2$,

$$\|S(a)\|_{L_p} \leqslant CK_1\left(\sqrt{p}\,\|a\|_2 + p\,\|a\|_\infty\right). \tag{42}$$

Solution:
Let $\alpha := K_1^2 \|a\|_2^2$ and $\beta := K_1 \|a\|_\infty$. Using the tail integral identity and bound in (41) gives

$$\mathbb{E}[|S(a)|^p] \leqslant p\int_0^\infty t^{p-1} \cdot 2\exp\left[-c\min\left(\frac{t^2}{\alpha}, \frac{t}{\beta}\right)\right] dt.$$

Using $e^{-\min\{u,v\}} \leqslant e^{-u} + e^{-v}$, we split the integral:

$$\frac{\mathbb{E}[|S(a)|^p]}{2pe^c} \leqslant \int_0^\infty t^{p-1} e^{-t^2/\alpha} dt + \int_0^\infty t^{p-1} e^{-t/\beta} dt = \frac{1}{2}\int_0^\infty z^{p/2-1} e^{-z/\alpha} dz + \int_0^\infty t^{p-1} e^{-t/\beta} dt,$$

where we do $z = t^2$ substitution in the first integral. Now, we recognize both integrals as the integrals of a Gamma kernel, which gives

$$\frac{\mathbb{E}[|S(a)|^p]}{2pe^c} \leqslant \frac{1}{2}\alpha^{p/2}\Gamma(p/2) + \beta^p \Gamma(p).$$

Rearranging the terms:

$$\mathbb{E}[|S(a)|^p] \leqslant pe^c K_1^p \left(\|a\|_2^p \Gamma(p/2) + 2\|a\|_\infty^p \Gamma(p)\right).$$

Using the bound $\Gamma(s) \leqslant s^s$ for $s \geqslant 1$ gives $\Gamma(p/2) \leqslant (\sqrt{p})^p$ and $\Gamma(p) \leqslant p^p$. So,

$$\left(\mathbb{E}[|S(a)|^p]\right)^{1/p} \leqslant C_1 p^{1/p} K_1 \left(\|a\|_2^p (\sqrt{p})^p + \|a\|_\infty^p p^p\right)^{1/p} \leqslant CK_1\left(\|a\|_2^p (\sqrt{p})^p + \|a\|_\infty^p p^p\right)^{1/p},$$

for some absolute constant $C > 0$ because $p^{1/p} \leqslant e^{1/e}$. Finally, using Minkowski's inequality, there exists an absolute constant $C > 0$ such that for every $p \geqslant 2$,

$$\|S(a)\|_{L_p} = \left(\mathbb{E}[|S(a)|^p]\right)^{1/p} \leqslant CK_1\left(\sqrt{p}\,\|a\|_2 + p\,\|a\|_\infty\right). \quad \square$$

(d) **[Bonus] The $p\|a\|_\infty$ term is unavoidable.** Give an example showing that the bound (42) cannot, in general, be improved to $\|S(a)\|_{L_p} \lesssim K_1\sqrt{p}\,\|a\|_2$ for all p.

[Bonus] Solution:
Let $N = 1, a_1 = 1$ and X_1 be centered Laplace (double-exponential) distribution with scale parameter 1, then here $S(a) = X_1$. The density of X_1 is given by:

$$f(x) = \frac{1}{2}e^{-|x|}, \qquad x \in \mathbb{R}.$$

We know X_1 is sub-Exponential so $\|X_1\|_{\psi_1} \asymp 1$. Now, for $p \geqslant 1$,

$$\mathbb{E}[|X_1|^p] = 2\int_0^\infty x^p \cdot \frac{1}{2}e^{-x}dx = \int_0^\infty x^p e^{-x}dx = \Gamma(p+1),$$

recognizing the integral of a Gamma kernel. Using Striling's approximation, $\Gamma(p+1) = p! \sim \sqrt{2\pi p}\left(\frac{p}{e}\right)^p$ so $(\Gamma(p+1))^{1/p} \sim \frac{p}{e}$. This gives

$$\|S(a)\|_{L_p} = \|X_1\|_{L_p} = (\Gamma(p+1))^{1/p} \asymp p \quad \text{for large } p.$$

Hence, the bound (42) cannot, in general, be improved to

$$\|S(a)\|_{L_p} \lesssim \sqrt{p} \quad \text{for all } p. \quad \square$$

5 Bounding quadratic forms (Hanson–Wright inequalities)

Let $g = (g_1, \ldots, g_n) \sim \mathcal{N}(0, I_n)$ have independent standard normal coordinates, and let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with zero diagonal ($a_{ii} = 0$ for all i). Define the second-order Gaussian chaos

$$Z := g^\top A g = \sum_{i,j=1}^{n} a_{ij} g_i g_j = 2 \sum_{1\leqslant i\leqslant j\leqslant n} a_{ij} g_i g_j.$$

Let $\|\cdot\|_{\mathrm{op}}$ denote the spectral/operator norm and $\|\cdot\|_{\mathrm{F}}$ the Frobenius norm:

$$\|A\|_{\mathrm{op}} := \sup_{\|x\|_2=1} \|Ax\|_2, \qquad \|A\|_{\mathrm{F}} := \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}.$$

(For symmetric A, $\|A\|_{\mathrm{op}}$ is the largest absolute eigenvalue, and $\|A\|_{\mathrm{F}}^2$ is the sum of squared eigenvalues.)

(a) **Mean and variance.** Compute $\mathbb{E}Z$ and $\mathrm{Var}Z$ explicitly, and show that

$$\mathbb{E}Z = 0, \qquad \mathrm{Var}(Z) = 2\|A\|_{\mathrm{F}}^2.$$

Solution:
Here, $\mathbb{E}[g_i g_j] = 0$ for all $i \neq j$ because $g_i \perp\!\!\!\perp g_j$ for all $i \neq j$, and $\mathbb{E}[g_i] = 0\ \forall\, i$. Then,

$$\begin{aligned}\mathbb{E}Z &= \mathbb{E}\left[\sum_{i,j=1}^{n} a_{ij} g_i g_j\right] \\ &= \underbrace{\mathbb{E}\left[\sum_{i=j} a_{ii} g_i g_j\right]}_{=0 \text{ b/c } a_{ii}=0\,\forall\, i} + \underbrace{\mathbb{E}\left[\sum_{i\neq j}^{n} a_{ij} g_i g_j\right]}_{\mathbb{E}[g_i g_j]=0\,\forall\, i\neq j} \\ &= 0. \quad \square\end{aligned}$$

We know, $\mathrm{Var}(Z) = \mathbb{E}Z^2 - (\mathbb{E}Z)^2 = \mathbb{E}Z^2$ because $\mathbb{E}Z = 0$ and $Z^2 = 4\sum_{i<j}\sum_{k<l} a_{ij} a_{kl} g_i g_j g_k g_l$. So,

$$\mathrm{Var}(Z) = \mathbb{E}\left[4\sum_{i<j}\sum_{k<l} a_{ij} a_{kl} g_i g_j g_k g_l\right] = 4\sum_{i<j}\sum_{k<l} a_{ij} a_{kl}\, \mathbb{E}[g_i g_j g_k g_l].$$

Here, $\mathbb{E}[g_i g_j g_k g_l] = 0$ for all the $(i,j) \neq (k,l)$ pairs because it will have at least one unpaired independent Gaussian and the whole expectation will go to 0. For $(i,j) = (k,l)$ pairs, $\mathbb{E}[g_i g_j g_k g_l] = \mathbb{E}[g_i^2]\mathbb{E}[g_j^2] = 1$. And, for a symmetric matrix A with $\mathrm{diag}(A) = 0$, $\|A\|_{\mathrm{F}}^2 = 2\sum_{i<j} a_{ij}^2$. Hence,

$$\mathrm{Var}(Z) = 4\sum_{i<j}\sum_{k<l} a_{ij} a_{kl}\, \mathbb{E}[g_i g_j g_k g_l] = 4\sum_{i<j} a_{ij}^2 = 2\left(2\sum_{i<j} a_{ij}^2\right) = 2\|A\|_{\mathrm{F}}^2. \quad \square$$

(b) **Variance bound using Gaussian Poincaré.** Let $f(x) = x^\top A x$. Use the Gaussian Poincaré inequality

$$\mathrm{Var}(f(g)) \leqslant \mathbb{E}\|\nabla f(g)\|_2^2$$

to prove that $\mathrm{Var}(Z) \leqslant 4\|A\|_F^2$. Compare this bound with your exact formula from part (a). Where did the extra factor come from?

Solution:
For $f(x) = x^\top Ax$, $\nabla f(x) = Ax + A^\top x = 2Ax$, because $A^\top = A$ for a symmetric matrix. Then,

$$\|\nabla f(g)\|_2^2 = \|2Ag\|_2^2 = 4(Ag)^\top(Ag) = 4g^\top A^2 g.$$

Taking the expectation gives,

$$\mathbb{E}\,\|\nabla f(g)\|_2^2 = 4\mathbb{E}\big[g^\top A^2 g\big] = 4\cdot\text{trace}(A^2) = 4\,\|A\|_{\text{F}}^2\,.$$

Since, g is Gaussian, using Gaussian Poincaré inequality gives,

$$\text{Var}(f(g)) \leqslant \mathbb{E}\,\|\nabla f(g)\|_2^2 = 4\,\|A\|_{\text{F}}^2\,. \quad \square$$

Comparing to the exact formula from (a), the Poincaré inequality is off by a factor of 2. For a quadratic f, the gradient grows linearly in g, and the square of that gives a factor of 4. However, the true variance depends on the fourth Gaussian moment. So, Poincaré is a first-order inequality while quadratic functions fluctuate according to second-order effects, and thus we get an extra factor of 2. Also, the Gaussian Poincaré inequality is a general-purpose bound for any differentiable function. It doesn't "know" that $f(x)$ is a quadratic form or that A has a zero diagonal, which ends up giving looser bound. □

(c) **Diagonalization and distributional representation.** Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Show that

$$Z \overset{d}{=} \sum_{i=1}^n \lambda_i(g_i^2 - 1).$$

Solution:
A is a symmetric matrix, so there exists an orthogonal matrix U such that $A = U^\top \Lambda U$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let $y = Ug$ then $\mathbb{E}[y] = 0$ and $\text{Var}(y) = U^\top \text{Var}(g) U = U^\top U = I_n$, which gives rotational invariance, i.e., $y \sim \mathcal{N}(0, I_n)$. Now,

$$Z = g^\top Ag = g^\top U^\top \Lambda U g = y^\top \Lambda y = \sum_{i=1}^n \lambda_i y_i^2.$$

Here, $y^\top y = g^\top U^\top U g = g^\top g \implies y_i^2 \overset{d}{=} g_i^2\ \forall i$, and $\text{trace}(A) = \sum_i \lambda_i = 0$. Hence,

$$Z \overset{d}{=} \sum_{i=1}^n \lambda_i g_i^2 - \sum_{i=1}^n \lambda_i \overset{d}{=} \sum_{i=1}^n \lambda_i(g_i^2 - 1). \quad \square$$

(d) **CGF of a centered χ_1^2.** Let $G \sim \mathcal{N}(0,1)$ and define $Y := G^2 - 1$. Show that for every $\theta < \frac{1}{2}$,

$$\log \mathbb{E} e^{\theta Y} = -\theta - \frac{1}{2}\log(1 - 2\theta).$$

Then prove the bound

$$\log \mathbb{E} e^{\theta Y} \leqslant \frac{\theta^2}{1 - (2\theta)_+} \qquad \text{for all } \theta < \frac{1}{2},$$

where $(u)_+ := \max\{u, 0\}$.

Solution:
Here, $\mathbb{E}e^{\theta Y} = \mathbb{E}\exp\big((G^2-1)\theta\big) = e^{-\theta}\mathbb{E}e^{\theta G^2}$. Now, using the definition of expectation:

$$\mathbb{E}e^{\theta Y} = e^{-\theta}\int_{-\infty}^{\infty} e^{\theta x^2}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \frac{e^{-\theta}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(1-2\theta)}dx,$$

where the integral converges iff $1-2\theta > 0$, i.e, $\theta < 1/2$. Recognizing the Gaussian integral with variance $\sigma^2 = (1-2\theta)^{-1}$ gives:

$$\mathbb{E}e^{\theta Y} = \frac{e^{-\theta}}{\sqrt{2\pi}}\cdot\sqrt{2\pi}\cdot(1-2\theta)^{-1/2} = e^{-\theta}(1-2\theta)^{-1/2}. \tag{43}$$

Finally, taking logarithm of (46) gives the exact CGF,

$$\log\mathbb{E}e^{\theta Y} = -\theta - \frac{1}{2}\log(1-2\theta), \qquad \forall\theta < \frac{1}{2}. \quad \square \tag{44}$$

We use Taylor expansion of $-\log(1-u) = \sum_{k=1}^{\infty}\frac{u^k}{k}$, valid for $|u| < 1$. When $\theta \geqslant 0$, let $u = 2\theta < 1$:

$$-\frac{1}{2}\log(1-2\theta) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2\theta)^k}{k} = \theta + \sum_{k=2}^{\infty}\frac{2^{k-1}\theta^k}{k}.$$

Substituting it back into the exact CGF expression in (44):

$$\log\mathbb{E}e^{\theta Y} = -\theta + \left(\theta + \sum_{k=2}^{\infty}\frac{2^{k-1}\theta^k}{k}\right) = \sum_{k=2}^{\infty}\frac{2^{k-1}\theta^k}{k} \leqslant \sum_{k=2}^{\infty}\frac{2^{k-1}\theta^k}{2} = \theta^2\sum_{k=0}^{\infty}(2\theta)^k = \frac{\theta^2}{1-2\theta},$$

where the last infinite sum is a Geometric series with ratio $2\theta < 1$. When $\theta < 0$, let $x = -2\theta > 0$ and substitute $x = -2\theta$ in equation (44), which gives $\log\mathbb{E}e^{\theta Y} \leqslant \big(x - \log(1+x)\big)/2$. We know,

$$\log(1+x) \geqslant x - \frac{x^2}{2} \iff x - \log(1+x) \leqslant \frac{x^2}{2}, \quad \text{for all } x \geqslant 0.$$

Therefore,

$$\log\mathbb{E}e^{\theta Y} \leqslant \frac{1}{2}\cdot\frac{x^2}{2} = \frac{1}{4}\cdot 4\theta^2 = \theta^2, \quad (\theta < 0)$$

Hence, we get the bound:

$$\log\mathbb{E}e^{\theta Y} \leqslant \frac{\theta^2}{1-(2\theta)_+}, \qquad \forall\theta < \frac{1}{2}. \quad \square \tag{45}$$

(e) **[Bonus] A Bernstein-type CGF bound.** Using parts (c) and (d) plus independence, show that for all λ satisfying

$$|\lambda| < \frac{1}{2\,\|A\|_{\mathrm{op}}},$$

the log-MGF of Z satisfies the "Bernstein form"

$$\log\mathbb{E}e^{\lambda Z} \leqslant \frac{\lambda^2\,\|A\|_F^2}{1-2|\lambda|\,\|A\|_{\mathrm{op}}}. \tag{46}$$

[Bonus] Solution:
Using the expression for Z from part (c):

$$\log \mathbb{E}e^{\theta Z} = \log \mathbb{E} \exp\left(\sum_{i=1}^{n} \lambda_i (g_i^2 - 1)\right) = \sum_{i=1}^{n} \log \mathbb{E} e^{\lambda\lambda_i (g_i^2 - 1)}, \tag{47}$$

because the g_i's are independent, so the expectation of the product will be the product of the expectations, and taking the logarithm results in the sum of log expectations. Here, each $g_i \sim \mathcal{N}(0,1)$ and let $\theta = \lambda\lambda_i$ then $|\lambda\lambda_i| < 1/2 \implies \theta < 1/2$. So, $|\lambda\lambda_i| \leqslant |\lambda| \|A\|_{\mathrm{op}} < 1/2 \implies |\lambda| < \dfrac{1}{2\|A\|_{\mathrm{op}}}$ suffices $\theta < 1/2$. Using the bound (45) from part (d) in (47), for all $|\lambda| < \dfrac{1}{2\|A\|_{\mathrm{op}}}$,

$$\log \mathbb{E}e^{\theta Z} \leqslant \sum_{i=1}^{n} \frac{(\lambda\lambda_i)^2}{1 - (2\lambda\lambda_i)_+}.$$

We know $(\lambda\lambda_i)_+ \leqslant |\lambda\lambda_i| \leqslant |\lambda| \|A\|_{\mathrm{op}}$ and $\|A\|_{\mathrm{F}}^2 = \sum_{i=1}^{n} \lambda_i^2$. Hence, for all $|\lambda| < \dfrac{1}{2\|A\|_{\mathrm{op}}}$,

$$\log \mathbb{E}e^{\theta Z} \leqslant \sum_{i=1}^{n} \frac{(\lambda\lambda_i)^2}{1 - 2|\lambda| \|A\|_{\mathrm{op}}} = \frac{\lambda^2}{1 - 2|\lambda| \|A\|_{\mathrm{op}}} \sum_{i=1}^{n} \lambda_i^2 = \frac{\lambda^2 \|A\|_{\mathrm{F}}^2}{1 - 2|\lambda| \|A\|_{\mathrm{op}}}. \quad \square$$

(f) **[Bonus] Tail bound.** Use the Chernoff bound together with (46) to prove that for all $u \geqslant 0$,

$$\mathbb{P}\left\{Z \geqslant 2\|A\|_{\mathrm{F}} \sqrt{u} + 2\|A\|_{\mathrm{op}} u\right\} \leqslant e^{-u}.$$

Deduce the more "Bernstein-looking" bound: for all $t \geqslant 0$,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \exp\left(-\frac{t^2/4}{\|A\|_{\mathrm{F}}^2 + \|A\|_{\mathrm{op}} t}\right).$$

Finally, show how to turn this into a two-sided bound for $|Z|$.

[Bonus] Solution:
Using Chernoff and (46), noting $|\lambda| = \lambda$ for $\lambda \geqslant 0$, on upper tail gives

$$\mathbb{P}\{Z \geqslant t\} \leqslant \inf_{\lambda > 0} \exp\left(-\lambda t + \log \mathbb{E}e^{\theta Z}\right) \leqslant \inf_{0 < \lambda < \frac{1}{2\|A\|_{\mathrm{op}}}} \exp\left(-\lambda t + \frac{\lambda^2 \|A\|_{\mathrm{F}}^2}{1 - 2\lambda \|A\|_{\mathrm{op}}}\right).$$

Choose $\lambda = \dfrac{\sqrt{u}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}}$, then $2\|A\|_{\mathrm{op}} \lambda = \dfrac{2\|A\|_{\mathrm{op}}\sqrt{u}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}} < 1$, so it is admissible. Then,

$$\begin{aligned}
-\lambda t + \frac{\lambda^2 \|A\|_{\mathrm{F}}^2}{1 - 2\lambda \|A\|_{\mathrm{op}}} &= \frac{-t\sqrt{u}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}} + \frac{u}{(\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u})^2} \cdot \frac{\|A\|_{\mathrm{F}}^2 (\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u})}{\|A\|_{\mathrm{F}}} \\
&= \frac{-t\sqrt{u}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}} + \frac{u\|A\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}} = \frac{-t\sqrt{u} + u\|A\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}} + 2\|A\|_{\mathrm{op}}\sqrt{u}}.
\end{aligned}$$

Substituting $t = 2\|A\|_F\sqrt{u} + 2\|A\|_{op}u = \sqrt{u}(2\|A\|_F + 2\|A\|_{op}\sqrt{u})$ gives,

$$-\lambda t + \frac{\lambda^2\|A\|_F^2}{1-2\lambda\|A\|_{op}} = -\frac{u(2\|A\|_F + 2\|A\|_{op}\sqrt{u} - \|A\|_F)}{\|A\|_F + 2 = \|A\|_{op}\sqrt{u}} = -u.$$

Hence, for all $u \geq 0$,

$$\mathbb{P}\left\{Z \geqslant 2\|A\|_F\sqrt{u} + 2\|A\|_{op}u\right\} \leqslant e^{-u}. \quad \square \tag{48}$$

Now, let $u = \dfrac{t^2/4}{\|A\|_F^2 + \|A\|_{op}t} \implies t^2 - 4\|A\|_{op}u\,t - 4\|A\|_F^2 u = 0$. Solving for $t \geqslant 0$ gives,

$$\begin{aligned} t = \frac{4\|A\|_{op}u + \sqrt{16\|A\|_{op}^2u^2 + 16\|A\|_F^2u}}{2} &= 2\|A\|_{op}u + 2\sqrt{\|A\|_{op}^2u^2 + \|A\|_F^2u} \\ &\geqslant 2\|A\|_{op}u + 2\sqrt{\|A\|_F^2u} \qquad [\because \|A\|_{op}^2u^2 \geqslant 0] \\ &= 2\|A\|_{op}u + 2\|A\|_F\sqrt{u}. \end{aligned}$$

Thus, the event $\{Z \geqslant t\} \subseteq \left\{Z \geqslant 2\|A\|_{op}u + 2\|A\|_F\sqrt{u}\right\}$. Hence, using (48), for all $t \geqslant 0$,

$$\mathbb{P}\{Z \geqslant t\} \leqslant \mathbb{P}\left\{Z \geqslant 2\|A\|_F\sqrt{u} + 2\|A\|_{op}u\right\} \leqslant e^{-u} = \exp\left(-\frac{t^2/4}{\|A\|_F^2 + \|A\|_{op}t}\right). \quad \square \tag{49}$$

Now, we apply the same argument for $-Z = g^\top(-A)g$. We know, $\|-A\|_F = \|A\|_F$ and $\|-A\|_{op} = \|A\|_{op}$, the same one-sided bound holds, i.e.,

$$\mathbb{P}\{Z \leqslant -t\} = \mathbb{P}\{Z \geqslant -t\} \leqslant \exp\left(-\frac{t^2/4}{\|A\|_F^2 + \|A\|_{op}t}\right).$$

Finally, using union bound we get, for all $t \geqslant 0$,

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{t^2/4}{\|A\|_F^2 + \|A\|_{op}t}\right). \quad \square \tag{50}$$

(g) **[Bonus] L_p bound.** Show that there is an absolute constant C such that for every integer $p \geqslant 2$,

$$\|Z\|_{L_p} \leqslant C\left(\sqrt{p}\|A\|_F + p\|A\|_{op}\right).$$

[Bonus] Solution:
Observe, for any $t \geqslant 0$,

$$\|A\|_F^2 + \|A\|_{op}t \leqslant 2\max\left(\|A\|_F^2, \|A\|_{op}t\right) \implies -\frac{t^2/4}{\|A\|_F^2 + \|A\|_{op}t} \leqslant -\frac{1}{8}\min\left(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|_{op}}\right).$$

So, the tail bound in (50) becomes

$$\mathbb{P}\{|Z| \geqslant t\} \leqslant 2\exp\left(-\frac{1}{8}\min\left(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|_{op}}\right)\right). \tag{51}$$

Now, using the tail integral identity,

$$\|Z\|_{L_p}^p = \mathbb{E}|Z|^p = p\int_0^\infty t^{p-1}\mathbb{P}\{|Z| \geqslant t\}\,dt.$$

Using the bound in (51),

$$\|Z\|_{L_p}^p \leqslant 2p\int_0^\infty t^{p-1}\exp\left(-\frac{1}{8}\min\left(\frac{t^2}{\|A\|_{\mathrm{F}}^2}, \frac{t}{\|A\|_{\mathrm{op}}}\right)\right)dt.$$

Using $e^{-\min\{u,v\}} \leqslant e^{-u} + e^{-v}$, we split the integral:

$$\|Z\|_{L_p}^p \leqslant 2p\int_0^\infty t^{p-1}\exp\left(-\frac{t^2}{8\,\|A\|_{\mathrm{F}}^2}\right)dt + 2p\int_0^\infty t^{p-1}\exp\left(-\frac{t}{8\,\|A\|_{\mathrm{op}}}\right)dt. \tag{52}$$

Using the Gamma-type estimates (for $\alpha > 0$):

$$\int_0^\infty t^{p-1}e^{-\alpha t^2}dt = \frac{1}{2}\alpha^{-p/2}\Gamma\left(\frac{p}{2}\right), \quad \int_0^\infty t^{p-1}e^{-\alpha t}dt = \alpha^{-p}\Gamma(p),$$

in (52), we get,

$$\|Z\|_{L_p}^p \leqslant 2p\cdot\frac{1}{2}(8\,\|A\|_{\mathrm{F}}^2)^{p/2}\Gamma\left(\frac{p}{2}\right) + 2p\cdot(8\,\|A\|_{\mathrm{op}})^p\Gamma(p) = p8^{p/2}\,\|A\|_{\mathrm{F}}^p\,\Gamma\left(\frac{p}{2}\right) + 2p(8\,\|A\|_{\mathrm{op}})^p\Gamma(p).$$

For $p \geqslant 2$, using the crude bounds $\Gamma(p/2) \leqslant (p/2)^{p/2}$ and $\Gamma(p) \leqslant p^p$,

$$\|Z\|_{L_p}^p \leqslant p8^{p/2}\,\|A\|_{\mathrm{F}}^p\left(\frac{p}{2}\right)^{p/2} + 2p(8\,\|A\|_{\mathrm{op}})^p p^p.$$

Using Minkowski's inequality,

$$\|Z\|_{L_p} \leqslant \left[p8^{p/2}\,\|A\|_{\mathrm{F}}^p\left(\frac{p}{2}\right)^{p/2}\right]^{1/p} + \left[2p(8\,\|A\|_{\mathrm{op}})^p p^p\right]^{1/p} \leqslant C_1 p^{1/p}\,\|A\|_{\mathrm{F}}\sqrt{p} + C_2 p^{1/p}\,\|A\|_{\mathrm{op}}\,p,$$

for some constants $C_1, C_2 > 0$. We know, $p^{1/p} \leqslant e^{1/e}$, so

$$\|Z\|_{L_p} \leqslant C_1'\,\|A\|_{\mathrm{F}}\sqrt{p} + C_2'\,\|A\|_{\mathrm{op}}\,p \leqslant \max\{C_1', C_2'\}\left(\sqrt{p}\,\|A\|_{\mathrm{F}} + p\,\|A\|_{\mathrm{op}}\right).$$

Hence, there is an absolute constant C such that for every integer $p \geqslant 2$,

$$\|Z\|_{L_p} \leqslant C\left(\sqrt{p}\,\|A\|_{\mathrm{F}} + p\,\|A\|_{\mathrm{op}}\right). \quad \square$$

* * * END OF SOLUTIONS * * *