Homework 3

SDS 391P.6, Spring 2026

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This homework is (also) a work in progress and is provided as-is for instructional purposes. The problems are drawn from various sources and do not (yet) have sufficient references to the original material. Additionally, errors may be present. so some cauution is advised! The document will be updated if corrections are necessary. Last updated: 2026-02-15.

0 Guidelines

- Please start early. If you have questions about the statements, notation, or possible typos, email us as soon as possible. When emailing about the course, please begin the subject line with [SDS 391P.6].
- Please begin your answer to each *main* question on a separate page. If you use any code, include it in an appendix. Submit a single combined PDF to Canvas. (If you encounter any submission issues, please let us know.)
- The problems and motivations draw on multiple sources (past course material, textbooks, and other standard references). If you use any external resources that materially guide your solution (beyond routine lookups), please cite them in your write-up.
- These questions are designed to build intuition and technique. You are welcome to go beyond what is explicitly asked. If you introduce additional assumptions (while keeping the spirit of the problem), state them clearly. If you discover something interesting along the way, feel free to include it as a brief remark; we may share especially instructive observations with the class.
- Parts labeled [Bonus] are optional: they are not required for full credit. They are intended as extra practice. You may skip them without penalty. If you attempt them, please label your solutions clearly with [Bonus].
- Many parts include hints intended to help you get started. You are not required to follow the suggested route, and you are encouraged to try alternative approaches when appropriate.
- We will grade primarily for correctness and clear reasoning. Do not over-optimize for minor presentation details. The spirit of the homework is for you to learn something new!

1 More sub-Gaussian characterizations

In Homework 2, Exercise 5, we saw several equivalent (up to constants) characterizations of sub-Gaussianity: quadratic CGF bounds, Gaussian tails, moment growth, and exponential-square integrability. In this exercise, we develop two additional ways to recognize sub-Gaussian random variables.

Throughout, $C, c > 0$ denote absolute constants (that may change from line to line), and we write $A \lesssim B$ to mean $A \leqslant CB$.

Let X be a real-valued random variable, and define the ψ_2 (Orlicz) norm

$$\|X\|_{\psi_2} := \inf\Big\{K > 0 : \ \mathbb{E}\exp\Big(\frac{X^2}{K^2}\Big) \leqslant 2\Big\}.$$

Recall that X is sub-Gaussian if and only if $\|X\|_{\psi_2} < \infty$.

(a) **Sub-Gaussianity via (almost) Gaussian tail domination.** Let $g \sim \mathcal{N}(0,1)$. For two random variables U, V, define the relation

$$U \preceq V \quad \iff \quad \mathbb{P}\{|U| \geqslant t\} \leqslant 2\,\mathbb{P}\{|V| \geqslant t\} \ \text{ for all } t \geqslant 0.$$

Show that X is sub-Gaussian if and only if there exists $K > 0$ such that

$$X \preceq Kg. \tag{1}$$

More precisely:

(i) If $\|X\|_{\psi_2} \leqslant L$, prove that (1) holds with $K \leqslant CL$.

(*Hint:* From Homework 2, Exercise 5, $\|X\|_{\psi_2} \leqslant L$ implies a tail bound

$$\mathbb{P}\{|X| \geqslant t\} \leqslant 2\exp\Big(-c\,\frac{t^2}{L^2}\Big) \qquad (t \geqslant 0).$$

To compare to Gaussian tails, use the following lower bound (Mills ratio): for all $u > 0$,

$$\mathbb{P}\{g \geqslant u\} \ \geqslant \ \frac{\varphi(u)}{u + u^{-1}} \qquad \text{where} \qquad \varphi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}. \tag{2}$$

Therefore,

$$\mathbb{P}\{|g| \geqslant u\} = 2\mathbb{P}\{g \geqslant u\} \ \geqslant \ c\,\frac{1}{u+1}\,e^{-u^2/2} \qquad (u > 1/2),$$

after adjusting constants (handle $u \in [0, 1/2]$ separately by a crude constant lower bound).

Now set $u := t/K$. You want to choose K (as a constant multiple of L) so that for all $t \geqslant 0$,

$$2\exp\Big(-c\,\frac{t^2}{L^2}\Big) \ \leqslant \ 2\,\mathbb{P}\{|Kg| \geqslant t\} = 2\,\mathbb{P}\{|g| \geqslant t/K\}.$$

Use the lower bound $\mathbb{P}\{|g| \geqslant u\} \gtrsim (u+1)^{-1}e^{-u^2/2}$ and absorb the $(u+1)^{-1}$ factor into the Gaussian exponent via a crude inequality like $\log(u+1) \leqslant u^2/4 + 1$.)

(ii) Conversely, if (1) holds for some K, prove that $\|X\|_{\psi_2} \leqslant CK$.

(*Hint:* Tail domination gives, for all $t \geqslant 0$,

$$\mathbb{P}\{|X| \geqslant t\} \leqslant 2\mathbb{P}\{|Kg| \geqslant t\} = 2\mathbb{P}\{|g| \geqslant t/K\}.$$

Use the standard Gaussian upper tail bound $\mathbb{P}\{|g| \geqslant u\} \leqslant 2e^{-u^2/2}$ to get

$$\mathbb{P}\{|X| \geqslant t\} \leqslant 4\exp\Big(-\frac{t^2}{2K^2}\Big).$$

Now invoke the direction "Gaussian tail $\Rightarrow \psi_2$" from Homework 2, Exercise 5 (or redo quickly): from this tail bound, show $\mathbb{E}\exp(X^2/(cK^2)) \leqslant 2$ for some absolute c, hence $\|X\|_{\psi_2} \lesssim K$.)

(b) [Bonus] Show by example that part (a) can fail if the definition of $\preceq$ is strengthened by removing the factor 2, i.e., if one demands $\mathbb{P}\{|U| \geqslant t\} \leqslant \mathbb{P}\{|V| \geqslant t\}$ for all $t \geqslant 0$.

(*Hint:* Take $X = \varepsilon$ where $\varepsilon \in \{-1, +1\}$ is Rademacher. Then $\mathbb{P}\{|\varepsilon| \geqslant 1\} = 1$. For any finite K, $\mathbb{P}\{|Kg| \geqslant 1\} = \mathbb{P}\{|g| \geqslant 1/K\} < 1$, so $\mathbb{P}\{|\varepsilon| \geqslant 1\} \leqslant \mathbb{P}\{|Kg| \geqslant 1\}$ is impossible. Compare: with the factor 2 present, it *is* possible to dominate ε by a Gaussian: choose K so that $\mathbb{P}\{|g| \geqslant 1/K\} \geqslant 1/2$.)

(c) **[Bonus] A local MGF of X^2 characterization.** Show that X is sub-Gaussian if and only if there exists $K > 0$ such that

$$\mathbb{E}\exp\big(\lambda^2 X^2\big) \;\leqslant\; \exp(\lambda^2 K^2) \qquad \text{for all } |\lambda| \leqslant \frac{1}{K}. \tag{3}$$

More precisely:

(i) Assume $\|X\|_{\psi_2} \leqslant L$. Prove that (3) holds with $K \leqslant CL$.

(*Hint:* Let $U := X^2/L^2$, so $\mathbb{E}e^U \leqslant 2$. For $0 \leqslant \theta \leqslant 1$, use Hölder (or Lyapunov) to show

$$\mathbb{E}e^{\theta U} \leqslant (\mathbb{E}e^U)^\theta \leqslant 2^\theta.$$

Now take $\theta = \lambda^2 L^2$ (so $\theta \leqslant 1$ when $|\lambda| \leqslant 1/L$) to get

$$\mathbb{E}e^{\lambda^2 X^2} = \mathbb{E}e^{\theta U} \leqslant 2^\theta = \exp\big((\log 2)\lambda^2 L^2\big).$$

Choose K as a suitable constant multiple of L so that $(\log 2)L^2 \leqslant K^2$ and $1/K \leqslant 1/L$.)

(ii) Conversely, assume (3) holds for some $K > 0$. Prove that X is sub-Gaussian with $\|X\|_{\psi_2} \leqslant 2K$.

(*Hint:* Plug $\lambda = 1/(2K)$ into (3):

$$\mathbb{E}\exp\Big(\frac{X^2}{4K^2}\Big) \leqslant \exp\Big(\frac{1}{4}\Big) < 2.$$

By the definition of the ψ_2 norm, this implies $\|X\|_{\psi_2} \leqslant 2K$.)

2 Sub-exponential characterizations

For even more practice moving between different equivalent "exponentiality" formulations, this exercise proves several equivalent (up to universal constants) ways of expressing that a centered random variable has exponential-type tails. The most common viewpoints are: (i) exponential tail decay, (ii) linear moment growth, (iii) ψ_1 (Orlicz) exponential integrability, and (iv) local quadratic control of the centered log-MGF. This should feel similar in spirit to Homework 2, Exercise 5 (sub-Gaussian characterizations), except that the log-MGF control is only *local*.

Let Z be a real-valued random variable with $\mathbb{E}Z = 0$. Define the centered log-MGF

$$\psi_Z(\lambda) \;:=\; \log \mathbb{E}e^{\lambda Z} \in (-\infty, \infty] \qquad (\lambda \in \mathbb{R}),$$

and define the ψ_1 (Orlicz) norm

$$\|Z\|_{\psi_1} \;:=\; \inf\Big\{K>0:\; \mathbb{E}\exp\Big(\frac{|Z|}{K}\Big)\leqslant 2\Big\}.$$

(a) **Four sub-exponential properties.** Fix $K>0$. Consider the statements:

(i) Tail bound: For all $t \geqslant 0$,

$$\mathbb{P}\{|Z|\geqslant t\} \;\leqslant\; 2\exp\Big(-\frac{t}{K}\Big).$$

(ii) Moment growth: There exists an absolute constant $C>0$ such that for all $p \geqslant 1$,

$$\big(\mathbb{E}|Z|^p\big)^{1/p} \;\leqslant\; C\,K\,p.$$

(iii) ψ_1-integrability:

$$\mathbb{E}\exp\Big(\frac{|Z|}{K}\Big) \;\leqslant\; 2.$$

(iv) Local quadratic log-MGF: There exist absolute constants $c, C>0$ such that for all $|\lambda| \leqslant \frac{c}{K}$,

$$\psi_Z(\lambda) \;\leqslant\; C\,\lambda^2 K^2.$$

Our goal is to show that (i)–(iv) are equivalent up to universal constants (i.e., the parameter K may change by at most an absolute constant factor from one statement to another).

(a1) (i) $\Rightarrow$ (ii).

(*Hint:* You can use results from Homework 1, Exercise 4 directly.)

(a2) (ii) $\Rightarrow$ (iii).

(*Hint:* Use the power series $\mathbb{E}e^{|Z|/(K)} = \sum_{p\geqslant 0}\mathbb{E}|Z|^p/(K)^p p!$ and Stirling $p! \geqslant (p/e)^p$.)

(a3) (iii) $\Rightarrow$ (iv).

(*Hint:* Use $e^x \leqslant 1+x+\frac{x^2}{2}e^{|x|}$ with $x=\lambda Z$ and $\mathbb{E}Z=0$. Then bound $Z^2 e^{|\lambda||Z|}$ by a multiple of $K^2 e^{|Z|/K}$ when $|\lambda|\leqslant c/K$. Conclude $\mathbb{E}e^{\lambda Z} \leqslant 1 + C_1\lambda^2K^2 \leqslant \exp(C_1\lambda^2K^2)$.)

(a4) (iv) $\Rightarrow$ (i). Assume (iv). Show there exists an absolute constant $c>0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\{|Z|\geqslant t\} \;\leqslant\; 2\exp\Big(-c\,\min\Big(\frac{t^2}{K^2}, \frac{t}{K}\Big)\Big).$$

Deduce in particular (i) (after adjusting constants).

(*Hint:* Apply Chernoff with $\lambda \geqslant 0$: $\mathbb{P}\{Z \geqslant t\} \leqslant \exp(-\lambda t + \psi_Z(\lambda))$, but you must respect the constraint $|\lambda| \leqslant c/K$ from (iv). Optimize over $\lambda \in [0, c/K]$ by comparing $\lambda^\star \asymp t/K^2$ to c/K.)

(b) **[Bonus] Comparison with sub-Gaussianity.** Show that sub-Gaussian implies sub-exponential: if Z is σ^2-sub-Gaussian, then $\|Z\|_{\psi_1} \leqslant C\sigma$. Give an explicit example of a centered random variable that is sub-exponential but not sub-Gaussian (e.g. $Z = X - \mathbb{E}X$ with $X \sim \mathrm{Exp}(1)$), and justify both claims from tails.

3 Bounding ℓ_p norms of random vectors

This exercise provides practice bounding ℓ_p norms of random vectors with independent sub-Gaussian entries. We will derive (i) expectation bounds and (ii) high-probability bounds. We have already discussed special cases of $\|X\|_2$ (Euclidean norm) and $\|X\|_\infty$ (maximum) in the class.

Let $X_1, \ldots, X_n$ be independent real random variables with $\mathbb{E}X_i = 0$. Assume they are sub-Gaussian and set

$$K := \max_{1\leqslant i\leqslant n} \|X_i\|_{\psi_2}.$$

Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$. For $p \in [1, \infty)$, define

$$\|X\|_p := \Big(\sum_{i=1}^n |X_i|^p\Big)^{1/p}, \qquad \|X\|_\infty := \max_{1\leqslant i\leqslant n} |X_i|.$$

Throughout, you may freely use the standard sub-Gaussian consequences: there exist universal constants $c, C > 0$ such that for every i and every $q \geqslant 1$,

$$(\mathbb{E}|X_i|^q)^{1/q} \leqslant CK\sqrt{q}, \qquad \text{and} \qquad \mathbb{P}\{|X_i| \geqslant t\} \leqslant 2\exp\Big(-c\frac{t^2}{K^2}\Big) \quad (t \geqslant 0).$$

(a) **Expected ℓ_p norm (finite p).** Show that for every $p \in [1, \infty)$,

$$\mathbb{E}\|X\|_p \leqslant CK\sqrt{p}\, n^{1/p}.$$

(*Hint:* Use Jensen with the concave function $u \mapsto u^{1/p}$: $\mathbb{E}(\sum |X_i|^p)^{1/p} \leqslant (\sum \mathbb{E}|X_i|^p)^{1/p}$, then apply the moment bound.)

(b) **Expected ℓ_∞ norm.** Prove that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\Big\{\|X\|_\infty \geqslant CK\big(\sqrt{\log n} + t\big)\Big\} \leqslant 2e^{-ct^2}.$$

Deduce that

$$\mathbb{E}\|X\|_\infty \leqslant CK\sqrt{\log n}.$$

(*Hint:* Union bound: $\mathbb{P}\{\|X\|_\infty \geqslant u\} \leqslant \sum_{i=1}^n \mathbb{P}\{|X_i| \geqslant u\}$, then choose u of the form $CK(\sqrt{\log n} + t)$. For the expectation, use the tail integral identity $\mathbb{E}Y = \int_0^\infty \mathbb{P}\{Y \geqslant s\}\, ds$.)

(c) **Two regimes of expected ℓ_p norms.** Show that

$$\mathbb{E}\|X\|_p \leqslant \begin{cases} CK\sqrt{p}\, n^{1/p}, & 1 \leqslant p \leqslant \log n, \\ CK\sqrt{\log n}, & p \geqslant \log n, \end{cases}$$

up to universal constants.

(*Hint:* Combine part (a) with part (b) using the deterministic inequality $\|x\|_p \leqslant n^{1/p}\|x\|_\infty$ and the fact that $n^{1/p} \leqslant e$ when $p \geqslant \log n$ from Homework 1, Exercise 1.)

(d) **[Bonus] High-probability bound for ℓ_p norm when $p \geqslant \log n$.** Fix $p \in [\log n, \infty]$. Show that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\Big\{\|X\|_p \geqslant CK\big(\sqrt{\log n} + t\big)\Big\} \leqslant 2e^{-ct^2}.$$

(*Hint:* Use $\|X\|_p \leqslant n^{1/p}\|X\|_\infty$ and $n^{1/p} \leqslant e$.)

(e) **[Bonus] High-probability bound for ℓ_p norm when $p \leqslant \log n$.** Fix $p \in [1, \log n]$. Show that for every $q \geqslant p$,

$$\big(\mathbb{E}\|X\|_p^q\big)^{1/q} \;\leqslant\; CK\, n^{1/p}\sqrt{q}.$$

Deduce that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\Big\{\|X\|_p \geqslant CK\, n^{1/p}\big(\sqrt{p}+t\big)\Big\} \;\leqslant\; 2e^{-ct^2}.$$

(*Hint:* Write $\|X\|_p^q = (\sum |X_i|^p)^{q/p}$ and use the inequality $(\sum a_i)^r \leqslant n^{r-1}\sum a_i^r$ valid for $r \geqslant 1$ with $r = q/p$. Then apply the sub-Gaussian moment bound to $\mathbb{E}|X_i|^q$. For the tail, apply Markov's inequality with $q = p + t^2$ and note that $\sqrt{p+t^2} \leqslant \sqrt{p} + t$.)

(f) **[Bonus] Thin shell for $\|X\|_2$ (sharper than part (d)).** Assume in addition that $\mathbb{E}X_i^2 = 1$ for all i. Show that there exist universal constants $c, C > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\Big\{\big|\|X\|_2 - \sqrt{n}\big| \geqslant t\Big\} \;\leqslant\; 2\exp\Big(-c\,\frac{t^2}{K^4}\Big).$$

(*Hint:* Show that $X_i^2 - 1$ is sub-exponential with $\|X_i^2 - 1\|_{\psi_1} \leqslant CK^2$. Apply Bernstein's inequality to $n^{-1}\|X\|_2^2 - 1 = n^{-1}\sum_{i=1}^n (X_i^2 - 1)$. Then convert a deviation bound for $\|X\|_2^2$ into one for $\|X\|_2$ using the elementary implication $|\frac{\|X\|_2}{\sqrt{n}} - 1| \geqslant \delta \;\Rightarrow\; |\frac{\|X\|_2^2}{n} - 1| \geqslant \max\{\delta, \delta^2\}$.)

4 Bounding L_p norms of linear forms (Khintchine inequalities)

This exercise provides practice bounding L^p norms of linear combinations of independent random variables. For sub-Gaussian inputs, the upper bounds have a purely Gaussian $\sqrt{p}$ growth. For sub-exponential inputs, one inevitably sees a two-term bound with both $\sqrt{p}$ and p behavior, reflecting Bernstein's two-regime tails.

Throughout, let $X_1, \dots, X_N$ be independent real random variables and let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$. Write

$$S(a) := \sum_{i=1}^N a_i X_i, \qquad \|a\|_2 := \Big(\sum_{i=1}^N a_i^2\Big)^{1/2}, \qquad \|a\|_\infty := \max_{1 \leqslant i \leqslant N} |a_i|.$$

Throughout, $C, c > 0$ denote absolute constants (that may change from line to line).

(a) **Sub-Gaussian Khintchine L_p bound ($p \geqslant 2$).** Assume

$$\mathbb{E}X_i = 0, \qquad \mathbb{E}X_i^2 = 1, \qquad \|X_i\|_{\psi_2} \leqslant K_2 \quad \text{for all } i,$$

for some $K_2 \geqslant 1$. Show that for every $p \in [2, \infty)$,

$$\|a\|_2 \;\leqslant\; \|S(a)\|_{L^p} \;\leqslant\; C\,K_2\sqrt{p}\,\|a\|_2. \tag{4}$$

(*Hint:* The lower bound follows from monotonicity $\|S\|_{L^p} \geqslant \|S\|_{L^2}$ and the Pythagorean identity $\|S(a)\|_{L^2} = \|a\|_2$. For the upper bound, use that a_iX_i is sub-Gaussian with $\|a_iX_i\|_{\psi_2} \leqslant |a_i|\|X_i\|_{\psi_2}$, and that sums of independent sub-Gaussians satisfy $\|S(a)\|_{\psi_2} \leqslant CK_2\|a\|_2$. Then apply the standard implication $\|Y\|_{L^p} \leqslant C\sqrt{p}\,\|Y\|_{\psi_2}$.)

(b) **Sub-exponential Bernstein–Khintchine tail bound.** Assume instead that

$$\mathbb{E}X_i = 0, \qquad \|X_i\|_{\psi_1} \leqslant K_1 \quad \text{for all } i,$$

for some $K_1 > 0$. Show that there exists an absolute constant $c > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\{|S(a)| \geqslant t\} \;\leqslant\; 2\exp\left[-c\min\left(\frac{t^2}{K_1^2\|a\|_2^2},\; \frac{t}{K_1\|a\|_\infty}\right)\right]. \tag{5}$$

(*Hint:* Each a_iX_i is sub-exponential with $\|a_iX_i\|_{\psi_1} \leqslant |a_i|K_1$. Apply Bernstein's inequality for sums of independent sub-exponentials with $V = \sum_{i=1}^N \|a_iX_i\|_{\psi_1}^2 \leqslant K_1^2\|a\|_2^2$, and $B = \max_i \|a_iX_i\|_{\psi_1} \leqslant K_1\|a\|_\infty$.)

(c) **Sub-exponential Bernstein–Khintchine L_p bound ($p \geqslant 2$).** Use (5) to show that there exists an absolute constant $C > 0$ such that for every $p \geqslant 2$,

$$\|S(a)\|_{L^p} \;\leqslant\; C\,K_1\Big(\sqrt{p}\,\|a\|_2 + p\,\|a\|_\infty\Big). \tag{6}$$

(*Hint:* Use the tail bound from part (b) together with the tail integral identity $\mathbb{E}|S|^p = p\int_0^\infty t^{p-1}\mathbb{P}\{|S| \geqslant t\}\,dt$. Use $e^{-\min(u,v)} \leqslant e^{-u} + e^{-v}$ to split the integral into a "Gaussian part" and an "exponential part." Recall the Gamma-type estimates (for $\alpha > 0$):

$$\int_0^\infty t^{p-1}e^{-\alpha t^2}\,dt = \frac{1}{2}\,\alpha^{-p/2}\Gamma(p/2), \qquad \int_0^\infty t^{p-1}e^{-\alpha t}\,dt = \alpha^{-p}\Gamma(p),$$

and use crude bounds such as $\Gamma(s) \leqslant s^s$ for $s \geqslant 1$.)

(d) **[Bonus] The $p\|a\|_\infty$ term is unavoidable.** Give an example showing that the bound (6) cannot, in general, be improved to $\|S(a)\|_{L^p} \lesssim K_1\sqrt{p}\|a\|_2$ for all p.

(*Hint:* Take $N = 1$, $a_1 = 1$, and let X_1 be a centered Laplace (double-exponential) or centered exponential-type random variable. Then $\|X_1\|_{L^p} \asymp p$ for large p.)

5 Bounding quadratic forms (Hanson–Wright inequalities)

This exercise studies a canonical order-two U-statistic (a Gaussian"chaos") and shows how two matrix norms control its fluctuations. This is a special case of the Hanson–Wright inequality.

Let $g = (g_1, \dots, g_n) \sim \mathcal{N}(0, I_n)$ have independent standard normal coordinates, and let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with zero diagonal ($a_{ii} = 0$ for all i). Define the second-order Gaussian chaos

$$Z \;:=\; g^\top Ag \;=\; \sum_{i,j=1}^n a_{ij}\,g_ig_j \;=\; 2\sum_{1\leqslant i<j\leqslant n} a_{ij}\,g_ig_j.$$

Let $\|\cdot\|$ denote the spectral/operator norm and $\|\cdot\|_{\mathrm{F}}$ the Frobenius norm:

$$\|A\| := \sup_{\|x\|_2=1}\|Ax\|_2, \qquad \|A\|_{\mathrm{F}} := \Big(\sum_{i,j=1}^n a_{ij}^2\Big)^{1/2}.$$

(For symmetric A, $\|A\|$ is the largest absolute eigenvalue, and $\|A\|_{\mathrm{F}}^2$ is the sum of squared eigenvalues.)

(a) **Mean and variance.** Compute $\mathbb{E}Z$ and $\mathrm{Var}(Z)$ explicitly, and show that

$$\mathbb{E}Z = 0, \qquad \mathrm{Var}(Z) = 2\|A\|_{\mathrm{F}}^2.$$

(*Hints:* (i) $\mathbb{E}(g^\top A g) = \mathrm{tr}(A)$ and $\mathrm{tr}(A) = 0$ because $\mathrm{diag}(A) = 0$. (ii) You may diagonalize A (see part (c)), or use Wick's formula $\mathbb{E}[g_i g_j g_k g_\ell] = \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}$.)

(b) **Variance bound using Gaussian Poincaré.** Let $f(x) = x^\top A x$. Use the Gaussian Poincaré inequality

$$\mathrm{Var}(f(g)) \leqslant \mathbb{E}\|\nabla f(g)\|_2^2$$

to prove that $\mathrm{Var}(Z) \leqslant 4\|A\|_{\mathrm{F}}^2$. Compare this bound with your exact formula from part (a). Where did the extra factor come from?

(*Hint:* For symmetric A, $\nabla f(x) = 2Ax$.)

(c) **Diagonalization and distributional representation.** Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Show that

$$Z \overset{d}{=} \sum_{i=1}^n \lambda_i\,(g_i^2 - 1).$$

(*Hint:* Diagonalize $A = U^\top \Lambda U$ with U orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Use rotational invariance: if $y := Ug$, then $y \sim \mathcal{N}(0, I_n)$. Finally, use $\sum_i \lambda_i = \mathrm{tr}(A) = 0$.)

(d) **CGF of a centered χ_1^2.** Let $G \sim \mathcal{N}(0,1)$ and define $Y := G^2 - 1$. Show that for every $\theta < \frac{1}{2}$,

$$\log \mathbb{E}e^{\theta Y} = -\theta - \frac{1}{2}\log(1 - 2\theta).$$

Then prove the bound

$$\log \mathbb{E}e^{\theta Y} \leqslant \frac{\theta^2}{1 - (2\theta)_+} \qquad \text{for all } \theta < \tfrac{1}{2},$$

where $(u)_+ := \max\{u, 0\}$.

(*Hint:* For $\theta \geqslant 0$, set $x = 2\theta \in [0,1)$ and bound $-\log(1-x) - x$ by a multiple of $x^2/(1-x)$. For $\theta \leqslant 0$, you can use a cruder bound, e.g., $-\log(1-x) \leqslant x + x^2$ for small x.)

(e) **[Bonus] A Bernstein-type CGF bound.** Using parts (c) and (d) plus independence, show that for all λ satisfying

$$|\lambda| < \frac{1}{2\|A\|},$$

the log-MGF of Z satisfies the "Bernstein form"

$$\log \mathbb{E}e^{\lambda Z} \leqslant \frac{\lambda^2 \|A\|_{\mathrm{F}}^2}{1 - 2|\lambda|\|A\|}. \tag{7}$$

(*Hint:* From part (c), $\log \mathbb{E}e^{\lambda Z} = \sum_{i=1}^n \log \mathbb{E}e^{\lambda\lambda_i(g_i^2 - 1)}$. Apply part (d) with $\theta = \lambda\lambda_i$ and use $(\lambda\lambda_i)_+ \leqslant |\lambda|\,\|A\|$.)

(f) **[Bonus] Tail bound.** Use the Chernoff bound together with (7) to prove that for all $u \geqslant 0$,

$$\mathbb{P}\Big\{Z \geqslant 2\|A\|_{\mathrm{F}}\sqrt{u} + 2\|A\|\,u\Big\} \leqslant e^{-u}.$$

Deduce the more "Bernstein-looking" bound: for all $t \geqslant 0$,

$$\mathbb{P}\{Z \geqslant t\} \;\leqslant\; \exp\left(-\,\frac{t^2/4}{\|A\|_{\mathrm{F}}^2 + \|A\|\,t}\right).$$

Finally, show how to turn this into a two-sided bound for $|Z|$.

(*Hint:* (i) For two-sided, apply the same argument to $-Z$ (equivalently, replace A by $-A$) and union bound. (ii) To pass between the "$2\sqrt{u} + 2u$" form and the "$\frac{t^2}{\|A\|_{\mathrm{F}}^2 + \|A\| t}$" form, solve for u as a function of t, or use the implication $t \leqslant 2a\sqrt{u} + 2bu \Rightarrow u \gtrsim \min\{t^2/a^2,\ t/b\}$ with appropriate constants.)

(g) **[Bonus]** L_p **bound.** Show that there is an absolute constant C such that for every integer $p \geqslant 2$,

$$\|Z\|_{L_p} \;\leqslant\; C\Big(\sqrt{p}\,\|A\|_{\mathrm{F}} + p\,\|A\|\Big).$$

(*Hint:* Use the tail bound from part (f) together with the tail integral identity $\mathbb{E}|Z|^p = p\int_0^\infty t^{p-1}\mathbb{P}\{|Z| \geqslant t\}\,dt$ and compare to Gamma-function integrals and their estimates as in Exercise 4, part (c).)

Source material

Parts of this homework were inspired by exercises from Boucheron et al. (2013); Tropp (2023); van Handel (2016), in addition to the author's accumulated experience working on related topics.

References

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