

Mid Exam
SDS 391P.6, Spring 2026
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Due: Mar 16 (Monday)

0 Guidelines

- The exam duration is 90 minutes. There are two main problems, worth 10 points each. Roughly speaking, we expect each main problem to be doable in about 30 minutes, leaving about 15 minutes of buffer per problem.
- Parts labeled [Bonus] are optional and are not required for full credit.
- The exam is closed book. You may use one sheet of notes (front and back).
- We will provide loose white sheets in class. Please write your name and EID clearly on every page, and number your pages.
- Please begin each main problem on a new page. Show enough work that your reasoning can be followed.
- At the end of the exam, please staple your pages together. A stapler will be provided in the classroom. On the front page, please write the total number of pages on your front page.
- You may quote any result proved in lecture or homework, provided you state clearly what you are using.
- We will allow for post-exam completion. Any clearly labeled post-exam additions submitted by the deadline (listed above) may earn up to 60% of the listed points on the relevant subpart. The recorded score on each subpart will be the maximum of the in-class score and the post-exam score.

1 Rademacher series and processes

Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher random variables, i.e.,

$$\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}.$$

For each $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, define the associated *Rademacher series*

$$S_t := \sum_{i=1}^n \varepsilon_i t_i.$$

Let $T \subset \mathbb{R}^n$ be a finite set, and define the associated *Rademacher process* supremum

$$Z := \sup_{t \in T} S_t.$$

We will use the two parameters

$$\sigma^2 := \sup_{t \in T} \|t\|_2^2 = \sup_{t \in T} \sum_{i=1}^n t_i^2, \quad \sigma_\infty^2 := \sum_{i=1}^n \sup_{t \in T} t_i^2.$$

The quantity σ^2 is the natural variance scale of a *single* Rademacher series, while σ_∞^2 is the coordinatewise bounded-differences proxy one gets by controlling the supremum one coordinate at a time.

- (a) **A fixed Rademacher series.** In this part, you will show that a fixed Rademacher series is centered, has variance determined by the ℓ_2 norm of its coefficient vector, satisfies a sub-Gaussian cumulant generating function bound, and therefore enjoys Gaussian tails and Gaussian-like moment growth.

(a1) Show that

$$\mathbb{E}S_t = 0, \quad \text{Var}(S_t) = \|t\|_2^2.$$

(2 points)

(a2) Use part (a1) and Chebyshev's inequality to show that for every $u > 0$,

$$\mathbb{P}\{|S_t| \geq u\} \leq \frac{\|t\|_2^2}{u^2}.$$

This gives a polynomial tail bound.

Next, show that the cumulant generating function of S_t satisfies

$$\psi_t(\lambda) := \log \mathbb{E}e^{\lambda S_t} \leq \frac{\lambda^2 \|t\|_2^2}{2} \quad \text{for all } \lambda \in \mathbb{R}.$$

Deduce that for every $u \geq 0$,

$$\mathbb{P}\{|S_t| \geq u\} \leq 2 \exp\left(-\frac{u^2}{2\|t\|_2^2}\right).$$

Thus, a Rademacher series has Gaussian tails on the scale $\|t\|_2$.

(2 points)

(a3) For $p \geq 1$, write

$$\|S_t\|_{L^p} := (\mathbb{E}|S_t|^p)^{1/p}.$$

Deduce from part (a2) that for every $p \geq 2$,

$$\|S_t\|_{L^p} \leq C\sqrt{p}\|t\|_2$$

for an absolute constant C .

This is the upper Khintchine-type bound for a fixed Rademacher series.

(2 points)

(b) **A coordinatewise variance bound for the Rademacher process supremum.** We now turn from a fixed series S_t to the supremum

$$Z = \sup_{t \in T} S_t.$$

The point of this part is to see what happens if we control Z by changing one coordinate at a time.

(b1) Suppose $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n)$ differ in only one coordinate, say coordinate i . Show that

$$\left| \sup_{t \in T} \sum_{j=1}^n \varepsilon_j t_j - \sup_{t \in T} \sum_{j=1}^n \tilde{\varepsilon}_j t_j \right| \leq 2 \sup_{t \in T} |t_i|.$$

(2 points)

(b2) Use the bounded differences variance inequality to deduce that

$$\text{Var}(Z) \leq \sigma_\infty^2.$$

(2 points)

(c) **[Bonus] A sharper variance bound for the Rademacher process supremum.** The bound in part (b) can be far from optimal because it treats each coordinate separately. We now show that the supremum only costs a constant factor relative to the natural scale σ^2 .

Let $\varepsilon'_1, \dots, \varepsilon'_n$ be an independent copy of $\varepsilon_1, \dots, \varepsilon_n$. For each i , define

$$\varepsilon^{(i)} := (\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon'_i, \varepsilon_{i+1}, \dots, \varepsilon_n), \quad Z^{(i)} := \sup_{t \in T} \sum_{j=1}^n \varepsilon_j^{(i)} t_j.$$

(c1) Fix a deterministic tie-breaking rule, and let $t^* = t^*(\varepsilon) \in T$ be a measurable maximizer such that

$$Z = \sum_{j=1}^n \varepsilon_j t_j^*.$$

Show that for every i ,

$$(Z - Z^{(i)})_+ \leq |\varepsilon_i - \varepsilon'_i| |t_i^*|.$$

(1 point)

(c2) Show that

$$\mathbb{E}[(\varepsilon_i - \varepsilon'_i)^2] = 2.$$

Then use the one-sided Efron–Stein–Steele inequality

$$\text{Var}(Z) \leq \sum_{i=1}^n \mathbb{E}[(Z - Z^{(i)})_+^2]$$

to deduce that

$$\text{Var}(Z) \leq 2\sigma^2.$$

(1 point)

(c3) Give an example of a finite set $T \subset \mathbb{R}^n$ for which

$$\frac{\sigma_\infty^2}{\sigma^2} \asymp n,$$

so that the bonus bound in part (c2) improves the bound in part (b2) by a factor of order n .

(1 point)

(d) [**Take-home bonus**] **Two tail bounds for the process supremum.** This part is not part of the in-class portion of the midterm; you may leave it for the take-home portion.

(d1) Use part (b1) and McDiarmid’s inequality to show that for every $u \geq 0$,

$$\mathbb{P}\{|Z - \mathbb{E}Z| \geq u\} \leq 2 \exp\left(-\frac{u^2}{2\sigma_\infty^2}\right).$$

(1 point)

(d2) Now derive a sharper one-sided tail bound using the entropy method. For each i , define

$$Z_i := \inf_{\eta \in \{-1, +1\}} \sup_{t \in T} \left(\sum_{j \neq i} \varepsilon_j t_j + \eta t_i \right).$$

Show that

$$0 \leq Z - Z_i \leq 2|t_i^*|, \quad \text{and hence} \quad \sum_{i=1}^n (Z - Z_i)^2 \leq 4\sigma^2.$$

Use the one-sided entropy-method concentration to deduce that for every $u \geq 0$,

$$\mathbb{P}\{Z - \mathbb{E}Z \geq u\} \leq \exp\left(-\frac{u^2}{16\sigma^2}\right).$$

Finally, explain briefly, using the example from part (c3), why this can be much sharper than the McDiarmid bound from part (d1).

(1 point)

2 Gaussian quadratic forms

Let $g = (g_1, \dots, g_n) \sim \mathcal{N}(0, I_n)$ have independent standard normal coordinates, and let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with zero diagonal ($a_{ii} = 0$ for all i). Define

$$Z := g^\top A g = \sum_{i,j=1}^n a_{ij} g_i g_j = 2 \sum_{1 \leq i < j \leq n} a_{ij} g_i g_j.$$

This is a Gaussian quadratic form, often called a *second-order Gaussian chaos*.

Let $\|A\|$ denote the spectral norm and $\|A\|_F$ the Frobenius norm:

$$\|A\| := \sup_{\|x\|_2=1} \|Ax\|_2, \quad \|A\|_F := \left(\sum_{i,j=1}^n a_{ij}^2 \right)^{1/2}.$$

For symmetric A , let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of A . Recall that

$$\operatorname{tr}(A) = \sum_{i=1}^n \lambda_i, \quad \|A\|_F^2 = \sum_{i=1}^n \lambda_i^2 = \operatorname{tr}(A^2).$$

Since A has zero diagonal, $\operatorname{tr}(A) = 0$.

- (a) **Diagonalization, exact moments, and comparison with Gaussian Poincaré.** In this part, you will first rewrite Z in a diagonal basis, then use that representation to compute its mean and variance exactly, and finally compare the exact variance with the upper bound given by Gaussian Poincaré.

(a1) Show that

$$Z \stackrel{d}{=} \sum_{i=1}^n \lambda_i (h_i^2 - 1),$$

where $h = (h_1, \dots, h_n) \sim \mathcal{N}(0, I_n)$ has independent standard normal coordinates. It may help to diagonalize $A = U^\top \Lambda U$ where U is orthogonal and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, set $h := Ug$, and use rotational invariance of the Gaussian distribution.

(2 points)

(a2) Deduce that

$$\mathbb{E}Z = 0, \quad \operatorname{Var}(Z) = 2\|A\|_F^2.$$

It may help to know that if $h_i \sim \mathcal{N}(0, 1)$ then

$$\mathbb{E}[h_i^2] = 1, \quad \mathbb{E}[h_i^4] = 3, \quad \operatorname{Var}(h_i^2 - 1) = 2.$$

(2 points)

- (a3) Let $f(x) = x^\top A x$. Recall that for symmetric A , $\nabla f(x) = 2Ax$. Use the Gaussian Poincaré inequality

$$\operatorname{Var}(f(g)) \leq \mathbb{E}\|\nabla f(g)\|_2^2$$

to show that

$$\operatorname{Var}(Z) \leq 4\|A\|_F^2.$$

Briefly explain why this is weaker than the exact formula from part (a2). It may help to note that for any deterministic symmetric matrix M ,

$$\mathbb{E}[g^\top M g] = \text{tr}(M).$$

(2 points)

- (b) **A Bernstein-type CGF bound and an upper-tail estimate.** This part shows that the Gaussian quadratic form Z behaves like a sub-exponential random variable: its Frobenius norm $\|A\|_F$ controls the Gaussian-scale fluctuations, while the operator norm $\|A\|$ controls the large-deviation scale.

For the rest of the problem, you may use the following fact without proof: if $G \sim \mathcal{N}(0, 1)$ and

$$Y := G^2 - 1,$$

then for every $\theta < \frac{1}{2}$,

$$\log \mathbb{E} e^{\theta Y} = -\theta - \frac{1}{2} \log(1 - 2\theta) \leq \frac{\theta^2}{1 - (2\theta)_+}, \quad (u)_+ := \max\{u, 0\}.$$

- (b1) Use part (a1) and independence to show that for every $\lambda \in \mathbb{R}$ satisfying

$$2|\lambda| \|A\| < 1,$$

we have

$$\log \mathbb{E} e^{\lambda Z} \leq \frac{\lambda^2 \|A\|_F^2}{1 - 2|\lambda| \|A\|}.$$

(2 points)

- (b2) Assume $A \neq 0$. For $u \geq 0$, define

$$\lambda_u := \frac{\sqrt{u}}{\|A\|_F + 2\|A\|\sqrt{u}}.$$

Show that $0 \leq \lambda_u$ and $2\lambda_u \|A\| < 1$, and show by direct algebra that

$$-\lambda_u \left(2\|A\|_F \sqrt{u} + 2\|A\| u \right) + \frac{\lambda_u^2 \|A\|_F^2}{1 - 2\lambda_u \|A\|} = -u.$$

Use Chernoff's bound together with part (b1) to deduce that

$$\mathbb{P}\left\{ Z \geq 2\|A\|_F \sqrt{u} + 2\|A\| u \right\} \leq e^{-u} \quad \text{for all } u \geq 0.$$

(2 points)

- (c) **[Bonus] A two-sided Bernstein form.** Show that there exists an absolute constant $c > 0$ such that for all $t \geq 0$,

$$\mathbb{P}\{|Z| \geq t\} \leq 2 \exp\left(-c \min\left(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|}\right)\right).$$

(1 point)